Uniqueness of Kerr-Newman solution

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Abstract. We show that non-degenerate multiple black hole solution of Einstein-Maxwell equations in an asymptotically flat axisymmetric spacetime cannot be in stationary equilibrium. This extends the uniqueness of Kerr-Newman solution first proved by Bunting and Mazur in a much wider desirable class. Spin-spin interaction cannot hold the black hole aparts even with electromagnetic forces.

Key words. Black hole uniqueness theorems, Kerr-Newman solution.

1 Introduction

We generalize the method used in [1],[2] for the uniqueness problem of Kerr-Newman solution for $M^2 > a^2 + \mathfrak{e}^2 + \mathfrak{m}^2$. Here $\mathfrak{m}$ is the magnetic charge. For a single black hole case the results are due to Bunting [3] and Mazur [4] (also Carter [5]) using different techniques. Several extensions of these results have been obtained by Wells [6]. Regarding the possibility of multiple black holes including those in a vacuum spacetime several results are obtained by Weinstein [7, 8] (see also the review article by Beig and Chrusciel [9]), Neugebauer and Meinel [10], Chrusciel and Costa [11], Wong and Yu [12]. Wong and Yu does not need the axisymmetric assumption but assumed the solution to be close to Kerr-Newman solution in some sense. We consider only non-degenerate black hole boundary. Our technique involves tailoring suitably the spinorial proofs of the positive mass theorem of Schoen and Yau [13] due to Witten [14] and Bartnik [15].

Initially we suppose the spacetime metric is stationary and axisymmetric EM

black hole solution having the form

$$\overset{4}{g} = \overset{4}{g}_{ab} dx^a dx^b = -V dt^2 + 2W dt d\phi + X d\phi^2 + \overline{g} \tag{1}$$

Carter (Part II, [16]) showed that $\overline{g} = \Omega(d\rho^2 + dz^2)$ where ρ and z are conjugate harmonic functions. V, W, X and Ω are functions of ρ and z. Following Carter we take

$$W^2 + VX = \rho^2 \tag{2}$$

2 Ricci Curvature

We denote the 2-metric by $\overline{g} = \overset{4}{g}_{11}(dx^1)^2 + \overset{4}{g}_{22}(dx^2)^2 = \overset{4}{g}_{AB} dx^A dx^B$. We denote the induced 3-metric on a t = constant hypersurface by

$$\widehat{g} = \overline{g} + X d\phi^2 \tag{3}$$

Then $\det([\overset{4}{g}_{ab}]) = -\left(VX + W^2\right)\overline{g}_{11}\overline{g}_{22}$. Using Eq. (97) we get

$$[\overset{4}{g}{}^{ab}] = \begin{bmatrix} -X\rho^{-2} & 0 & 0 & W\rho^{-2} \\ 0 & \overline{g}_{11}^{-1} & 0 & 0 \\ 0 & 0 & \overline{g}_{22}^{-1} & 0 \\ W\rho^{-2} & 0 & 0 & V\rho^{-2} \end{bmatrix}$$

Henceforth $\overline{g}^{AB}$ is obtained from $\overline{g}_{AB}$ by raising the indices with the 2-metric $\overline{g}$. The t=constant surface has zero mean curvature. $\widehat{g}$ is a Riemannian metric. We shall denote the Laplacian and the covariant derivative of the two-metric $\overline{g}$ by $\overline{\Delta}$ and $\overline{\nabla}$. In general we shall use the metrics as subscripts in order to indicate w.r.t. which metric a norm or an operator is computed. Since we shall not use the usual formulations of a stationary axisymmetric vacuum spacetime it is better to give the expressions for the components of the Ricci curvature $\overset{4}{g}$ for easy reference before equating them to zero using the vacuum Einstein equations. Ricci curvature of the four metric is

$$\begin{aligned}
\overset{4}{R}_{tt} &= \frac{1}{2}\overline{\Delta}V + \frac{V}{4\rho^2}\left\langle \overline{\nabla}V, \overline{\nabla}X \right\rangle + \frac{V}{2\rho^2}|\overline{\nabla}W|^2 - \frac{W}{2\rho^2}\left\langle \overline{\nabla}V, \overline{\nabla}W \right\rangle - \frac{X}{4\rho^2}|\overline{\nabla}V|^2 && (4) \\
\overset{4}{R}_{t\phi} &= -\frac{1}{2}\overline{\Delta}W + \frac{V}{4\rho^2}\left\langle \overline{\nabla}W, \overline{\nabla}X \right\rangle - \frac{W}{2\rho^2}\left\langle \overline{\nabla}V, \overline{\nabla}X \right\rangle + \frac{X}{4\rho^2}\left\langle \overline{\nabla}V, \overline{\nabla}W \right\rangle && (5) \\
\overset{4}{R}_{\phi\phi} &= -\frac{1}{2}\overline{\Delta}X - \frac{X}{4\rho^2}\left\langle \overline{\nabla}V, \overline{\nabla}X \right\rangle + \frac{W}{2\rho^2}\left\langle \overline{\nabla}W, \overline{\nabla}X \right\rangle - \frac{X}{2\rho^2}|\overline{\nabla}W|^2 + \frac{V}{4\rho^2}|\overline{\nabla}X|^2 && (6) \\
R^{(4)}_{At} &= 0 = R^{(4)}_{A\phi} && (7) \\
\overset{4}{R}_{BD} &= \frac{1}{2}\overline{R}\overline{g}_{BD} - \frac{1}{\rho}\overline{\nabla}_D\overline{\nabla}_B\rho + \frac{1}{2\rho^2}\overline{\nabla}_D W\overline{\nabla}_B W + \frac{1}{4\rho^2}\overline{\nabla}_D V\overline{\nabla}_B X + \frac{1}{4\rho^2}\overline{\nabla}_B V\overline{\nabla}_D X && (8)
\end{aligned}$$

3 Einstein equation

$$\overset{4}{R}_{ab} = 8\pi\left(T_{ab} - (1/2)T\overset{4}{g}_{ab}\right) \tag{9}$$

with energy-momentum tensor

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_{bd}\overset{4}{g}^{cd} - (1/4)\overset{4}{g}_{ab}F_{ij}F_{kl}\overset{4}{g}^{ik}\overset{4}{g}^{jl}\right) \tag{10}$$

F is the electromagnetic field tensor obtained from the electromagnetic potential one form **A**

$$F_{ab} = \frac{\partial \mathbf{A}_b}{\partial x^a} - \frac{\partial \mathbf{A}_a}{\partial x^b} \tag{11}$$

Since $T = \overset{4}{g}^{ab}T_{ab} = 0$ Einstein equation becomes

$$\overset{4}{R}_{ab} = 8\pi T_{ab} \tag{12}$$

Following Carter (Eq. 7.43 Part II [16]) we choose the electromagnetic potential to be of the form

$$\mathbf{A} = \xi dt + \psi d\phi \tag{13}$$

where ψ is a function of x^1 and x^2. (Our sign in Eq. (11) is opposite to that of Eq.6.48 in Carter). Thus

One finds

$$T_{tt} = \frac{1}{8\pi}\left(|\overline{\nabla}\xi|^2 + \left|W\overline{\nabla}\xi + V\overline{\nabla}\psi\right|^2\rho^{-2}\right) \tag{14}$$

$$T_{t\phi} = \frac{1}{8\pi}\left(2VX\left\langle\overline{\nabla}\xi, \overline{\nabla}\psi\right\rangle\rho^{-2} + WX|\nabla\xi|^2\rho^{-2} - WV|\overline{\nabla}\psi|^2\rho^{-2}\right) \tag{15}$$

$$T_{\phi\phi} = \frac{1}{8\pi}\left(|\overline{\nabla}\psi|^2 + \left|X\overline{\nabla}\xi + W\overline{\nabla}\psi\right|^2\rho^{-2}\right) \tag{16}$$

$$T_{AB} = \frac{1}{4\pi\rho^2}\left(-X\frac{\partial\xi}{\partial x^A}\frac{\partial\xi}{\partial x^B} + W\frac{\partial\xi}{\partial x^A}\frac{\partial\psi}{\partial x^B} + W\frac{\partial\psi}{\partial x^A}\frac{\partial\xi}{\partial x^B} + V\frac{\partial\psi}{\partial x^A}\frac{\partial\psi}{\partial x^B}\right. \tag{17}$$

$$\left.+\frac{X}{2}|\overline{\nabla}\xi|^2\overline{g}_{AB} - W\left\langle\overline{\nabla}\xi, \overline{\nabla}\psi\right\rangle\overline{g}_{AB} - \frac{V}{2}|\overline{\nabla}\psi|^2\overline{g}_{AB}\right) \tag{18}$$

$$T_{At} = 0 = T_{A\phi} \tag{19}$$

Einstein equation Eq. (12) becomes

$$\overset{4}{R}_{tt} = |\overline{\nabla}\xi|^2 + \left|W\overline{\nabla}\xi + V\overline{\nabla}\psi\right|^2 \rho^{-2} \tag{20}$$

$$\overset{4}{R}_{t\phi} = 2VX\left\langle\overline{\nabla}\xi, \overline{\nabla}\psi\right\rangle\rho^{-2} + WX|\nabla\xi|^2\rho^{-2} - WV|\overline{\nabla}\psi|^2\rho^{-2} \tag{21}$$

$$\overset{4}{R}_{\phi\phi} = |\overline{\nabla}\psi|^2 + \left|X\overline{\nabla}\xi + W\overline{\nabla}\psi\right|^2 \rho^{-2} \tag{22}$$

$$\overset{4}{R}_{BD} = \frac{2}{\rho^2}\left(-X\frac{\partial\xi}{\partial x^B}\frac{\partial\xi}{\partial x^D} + W\frac{\partial\xi}{\partial x^B}\frac{\partial\psi}{\partial x^D} + W\frac{\partial\psi}{\partial x^B}\frac{\partial\xi}{\partial x^D} + V\frac{\partial\psi}{\partial x^B}\frac{\partial\psi}{\partial x^D}\right. \tag{23}$$

$$\left. + \frac{X}{2}|\overline{\nabla}\xi|^2\overline{g}_{BD} - W\left\langle\overline{\nabla}\xi, \overline{\nabla}\psi\right\rangle\overline{g}_{BD} - \frac{V}{2}|\overline{\nabla}\psi|^2\overline{g}_{BD}\right) \tag{24}$$

First three of the following five equations we get from Eqs. (4-6) and Eqs. (21-23) respectively. Last two equations are the non-trivial equations in the nontrivial set of Maxwell equations namely $F_{ab}\overset{4}{;}_{c}\overset{4}{g}^{bc} = 0$ for $a = 0$ and $a = 3$.

$$\overline{\Delta}V = -\frac{V}{2\rho^2}\left\langle\overline{\nabla}X, \overline{\nabla}V\right\rangle - \frac{V|\overline{\nabla}W|^2}{\rho^2} + \frac{W}{\rho^2}\left\langle\overline{\nabla}W, \overline{\nabla}V\right\rangle + \frac{X|\overline{\nabla}V|^2}{2\rho^2} + 2|\overline{\nabla}\xi|^2 + 2\left|W\overline{\nabla}\xi + V\overline{\nabla}\psi\right|^2\rho^{-2} \tag{25}$$

$$\overline{\Delta}W = \frac{V}{2\rho^2}\left\langle\overline{\nabla}X, \overline{\nabla}W\right\rangle - \frac{W}{\rho^2}\left\langle\overline{\nabla}V, \overline{\nabla}X\right\rangle + \frac{X}{2\rho^2}\left\langle\overline{\nabla}V, \overline{\nabla}W\right\rangle - \frac{4VX}{\rho^2}\left\langle\overline{\nabla}\xi, \overline{\nabla}\psi\right\rangle + \frac{2W}{\rho^2}\left(V|\overline{\nabla}\psi|^2 - X|\nabla\xi|^2\right) \tag{26}$$

$$\overline{\Delta}X = -\frac{X}{2\rho^2}\left\langle\overline{\nabla}V, \overline{\nabla}X\right\rangle + \frac{W}{\rho^2}\left\langle\overline{\nabla}W, \overline{\nabla}X\right\rangle - \frac{X|\overline{\nabla}W|^2}{\rho^2} + \frac{V|\overline{\nabla}X|^2}{2\rho^2} - 2|\overline{\nabla}\psi|^2 - 2\left|X\overline{\nabla}\xi + W\overline{\nabla}\psi\right|^2\rho^{-2} \tag{27}$$

$$\overline{\Delta}\xi = \frac{X}{2\rho^2}\left\langle\overline{\nabla}V, \overline{\nabla}\xi\right\rangle - \frac{V}{2\rho^2}\left\langle\overline{\nabla}X, \overline{\nabla}\xi\right\rangle + \frac{V}{\rho^2}\left\langle\overline{\nabla}W, \overline{\nabla}\psi\right\rangle - \frac{W}{\rho^2}\left\langle\overline{\nabla}V, \overline{\nabla}\psi\right\rangle \tag{28}$$

$$\overline{\Delta}\psi = -\frac{X}{2\rho^2}\left\langle\overline{\nabla}V, \overline{\nabla}\psi\right\rangle + \frac{V}{2\rho^2}\left\langle\overline{\nabla}X, \overline{\nabla}\psi\right\rangle - \frac{X}{\rho^2}\left\langle\overline{\nabla}W, \overline{\nabla}\xi\right\rangle + \frac{W}{\rho^2}\left\langle\overline{\nabla}X, \overline{\nabla}\xi\right\rangle \tag{29}$$

Last two equations are the same as in Bunting's thesis (replacing his functions E, F, A, B, C by $\xi, \psi, -V, X, W$) and are equivalent to the set given by Carter (p74, Part II [16]):

$$\overline{\nabla}\left(\left(X\overline{\nabla}\xi - W\overline{\nabla}\psi\right)\rho^{-1}\right) = 0$$

$$\overline{\nabla}\left(\rho X^{-1}\overline{\nabla}\psi + W(\rho X)^{-1}\left(X\overline{\nabla}\xi - W\overline{\nabla}\psi\right)\right) = 0$$

From Eqs. (25-27) one can show that $\rho = \sqrt{VX + W^2}$ is a harmonic function i.e. $\overline{\Delta}\rho = 0$. This we are assuming from the start.

4 Remaining equations

Eqs. (8,24) give

$$
\begin{aligned}
&\frac{1}{2}\overline{R}\overline{g}_{BD} - \frac{1}{\rho}\overline{\nabla}_D\overline{\nabla}_B\rho + \frac{1}{2\rho^2}\frac{\partial W}{\partial x^D}\frac{\partial W}{\partial x^B} + \frac{1}{4\rho^2}\frac{\partial V}{\partial x^D}\frac{\partial X}{\partial x^B} + \frac{1}{4\rho^2}\frac{\partial V}{\partial x^B}\frac{\partial X}{\partial x^D} \\
&= \frac{2}{\rho^2}\left(-X\frac{\partial \xi}{\partial x^B}\frac{\partial \xi}{\partial x^D} + W\frac{\partial \xi}{\partial x^B}\frac{\partial \psi}{\partial x^D} + W\frac{\partial \psi}{\partial x^B}\frac{\partial \xi}{\partial x^D} + V\frac{\partial \psi}{\partial x^B}\frac{\partial \psi}{\partial x^D} + \frac{X}{2}|\overline{\nabla}\xi|^2\overline{g}_{BD} - W\left\langle\overline{\nabla}\xi, \overline{\nabla}\psi\right\rangle\overline{g}_{BD} - \frac{V}{2}|\overline{\nabla}\psi|^2\overline{g}_{BD}\right)
\end{aligned}
\tag{30}
$$

Contracting we get

$$
\overline{R} = -\frac{1}{2\rho^2}|\overline{\nabla}W|^2 - \frac{1}{2\rho^2}\left\langle\overline{\nabla}V, \overline{\nabla}X\right\rangle \tag{31}
$$

Differentiating Eq. (97) we get $2\rho\overline{\nabla}\rho = V\overline{\nabla}X + X\overline{\nabla}V + 2W\overline{\nabla}W$ so that

$$
2\rho\left\langle\overline{\nabla}\rho, \overline{\nabla}V\right\rangle = V\left\langle\overline{\nabla}X, \overline{\nabla}V\right\rangle + 2W\left\langle\overline{\nabla}W, \overline{\nabla}V\right\rangle + X|\overline{\nabla}V|^2 \tag{32}
$$

$$
2\rho\left\langle\overline{\nabla}\rho, \overline{\nabla}X\right\rangle = X\left\langle\overline{\nabla}V, \overline{\nabla}X\right\rangle + 2W\left\langle\overline{\nabla}W, \overline{\nabla}X\right\rangle + V|\overline{\nabla}X|^2 \tag{33}
$$

$$
2\rho\left\langle\overline{\nabla}\rho, \overline{\nabla}W\right\rangle = 2W|\overline{\nabla}W|^2 + V\left\langle\overline{\nabla}X, \overline{\nabla}W\right\rangle + X\left\langle\overline{\nabla}V, \overline{\nabla}W\right\rangle \tag{34}
$$

Using the above relations and Eq. (31) we write Eqs. (25,27) as

$$
\overline{\Delta}V = 2\overline{R}V + \left\langle\overline{\nabla}\ln\rho, \overline{\nabla}V\right\rangle + 2|\overline{\nabla}\xi|^2 + 2\left|W\overline{\nabla}\xi + V\overline{\nabla}\psi\right|^2\rho^{-2} \tag{35}
$$

$$
\overline{\Delta}W = 2\overline{R}W + \left\langle\overline{\nabla}\ln\rho, \overline{\nabla}W\right\rangle - \frac{4VX}{\rho^2}\left\langle\overline{\nabla}\xi, \overline{\nabla}\psi\right\rangle + \frac{2W}{\rho^2}\left(V|\overline{\nabla}\psi|^2 - X|\nabla\xi|^2\right) \tag{36}
$$

$$
\overline{\Delta}X = 2\overline{R}X + \left\langle\overline{\nabla}\ln\rho, \overline{\nabla}X\right\rangle - 2|\overline{\nabla}\psi|^2 - 2\left|X\overline{\nabla}\xi + W\overline{\nabla}\psi\right|^2\rho^{-2} \tag{37}
$$

Since it is well-known that the equations for Ω can be solved using its asymptotic value once we know the other functions we do not include the complicated equations for it.

The t = constant hypersurface has the topology $\Sigma^+ \cup \partial\Sigma^+$ where Σ^+ is an open 3-manifold and the boundary $\partial\Sigma^+$ is a finite number of disconnected 2-spheres. $X > 0$ in Σ^+ except on the axis. $(\partial\Sigma^+, \overline{g})$ is a smooth totally geodesic submanifold of the 3-dimensional Riemannian manifold with boundary $(\Sigma^+ \cup \partial\Sigma^+, \widehat{g})$. the 3-metric $\widehat{g}$ has nonnegative scalar curvature, which is easy to see from Eq. (12), weak energy condition and doubly contracted Gauss equation for the maximal t = constant hypersurface.

Let $\varrho^2 = r^2 + a^2\cos^2\theta$ and

$$M' = M - \frac{\mathfrak{e}^2 + \mathfrak{m}^2}{2r} \tag{38}$$

Kerr-Newman solution has the spacetime metric,

$$ds^2 = -\left(1 - \frac{2M'r}{\varrho^2}\right)dt^2 - \frac{4M'ra\sin^2\theta}{\varrho^2}d\phi dt + \left((r^2+a^2)\sin^2\theta + \frac{2M'ra^2\sin^4\theta}{\varrho^2}\right)d\phi^2 + \varrho^2\left(\frac{dr^2}{r^2 - 2M'r + a^2} + d\theta^2\right) \tag{39}$$

Soon after Kerr's discovery [17], Kerr-Newman solution was found by Newman and *et al.* [18]. The form given above is obtained from Carter (Eq. 5.54, Part I, [16]) by collecting the terms containing $dt^2, d\phi^2, dtd\phi$. This form includes the magnetic charge which can be removed by a duality transformation without changing the metric because the sum $\mathfrak{e}^2 + \mathfrak{m}^2$ remains constant under a duality transformation of the electromagnetic fields. The electromagnetic potential is

$$\mathbf{A}_{\mathrm{K}} = -\frac{\mathfrak{e}r + \mathfrak{m}a\cos\theta}{\varrho^2}dt + \frac{\mathfrak{e}ar\sin^2\theta + \mathfrak{m}(r^2+a^2)\cos\theta}{\varrho^2}d\phi \tag{40}$$

We use subscript K for Kerr-Newman. Comparing with Eq. (1) we get

$$V_{\mathrm{K}} = 1 - \frac{2M'r}{\varrho^2} \tag{41}$$

$$W_{\mathrm{K}} = -\frac{2M'ra\sin^2\theta}{\varrho^2} \tag{42}$$

$$X_{\mathrm{K}} = (r^2+a^2)\sin^2\theta + \frac{2M'ra^2\sin^4\theta}{\varrho^2} \tag{43}$$

In the general spacetime under investigation which is not yet known to be Kerr-Newman solution we shall choose r, θ coordinates from Carter's ρ, z coordinates as follows. Let r, θ be solution of the following equations with $r \geq M + \sqrt{M^2 - \mathfrak{e}^2 - \mathfrak{m}^2 - a^2}$,

$$\rho = \sqrt{r^2 - 2M'r + a^2}\sin\theta, \qquad z = (r - M)\cos\theta \tag{44}$$

In the equation for z we use the constant M because $(\partial\rho/\partial r) = (r^2 - 2M'r + a^2)^{-1/2}(r - M)\sin\theta$. This way $d\rho^2 + dz^2$ does not have a cross term containing $drd\theta$. The expression for $d\rho^2 + dz^2$ is given in §6. $\rho = 0$ set which represents the horizon and the axis is now given by

$$r^2 - 2Mr + a^2 + \mathfrak{e}^2 + \mathfrak{m}^2 = 0, \text{ or } \sin\theta = 0 \tag{45}$$

For convenience we define

$$c^2 = M^2 - \mathfrak{e}^2 - \mathfrak{m}^2 - a^2, \qquad c > 0 \tag{46}$$

The restriction on r now becomes $r \geq M + c$. In general (r, θ) coordinate system is defined away from the $\rho = 0$ set although the functions r and θ are defined on this set. Because of the restriction $r \geq M + c$ the equality is the only solution of the first equation of Eq. (45). The limiting set $r \downarrow M + c$ now contains the horizon and possibly some parts of the axis while $r > M + c, 0 \leq \theta \leq \pi$ represent the remaining part of Σ^+. The portion of the axis in this remaining part of Σ^+ is later called the part of the axis given by $\theta = 0$ or $\theta = \pi$ "alone."

5 Main Idea

We define some quantities which are crucial for the proof.

$$\begin{aligned} 2\widetilde{r} &= r - M + \sqrt{r^2 - 2Mr + \mathfrak{e}^2 + \mathfrak{m}^2 + a^2} && (47) \\ \zeta &= \widetilde{r}^2 \varrho^{-2} && (48) \\ f &= \widetilde{r}^2 \sin^2\theta && (49) \end{aligned}$$

Significance of these quantities is that they transform the 2-metric

$$\overline{g}_{\mathrm{K}} = \varrho^2 \left(\frac{dr^2}{r^2 - 2M'r + a^2} + d\theta^2 \right) \tag{50}$$

into the Euclidean 3-metric η_{K} in the spherical coordinates $\{\widetilde{r}, \theta, \phi\}$ as follows

$$\eta_{\mathrm{K}} = \zeta \overline{g}_{\mathrm{K}} + f d\phi^2 = d\widetilde{r}^2 + \widetilde{r}^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \tag{51}$$

Our aim is to show, by exploiting the field equations (25-30) and reasonable boundary conditions, that the general 3-metric η defined by

$$\eta = \zeta \overline{g} + f d\phi^2 \tag{52}$$

where ζ and f are the same functions of (r, θ) is the same Euclidean metric in the coordinates $\{\widetilde{r}, \theta, \phi\}$ where $\widetilde{r}$ is the same function of r. In the actual process we get

$X = X_{\rm K}$ at first. Then we show $\Omega = \Omega_{\rm K}$. This gives a single black hole so that the uniqueness proof of Bunting or Mazur applies. Let

$$r_{\rm out} = \widetilde{r}, \qquad r_{\rm in} = (1/2)\left(r - M - \sqrt{r^2 - 2Mr + \mathfrak{e}^2 + \mathfrak{m}^2 + a^2}\right) \tag{53}$$

If we take ζ and f as ζ^+ and $f^+$and define

$$\zeta^- = r_{\rm in}^2 \varrho^{-2} \tag{54}$$

$$f^- = r_{\rm in}^2 \sin^2\theta \tag{55}$$

then $\eta_{\rm K}^- = \zeta^- \overline{g}_{\rm K} + f^- d\phi^2 = dr_{\rm in}^2 + r_{\rm in}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$ is also the Euclidean metric. Recalling Eq. (46) we note that $r_{\rm in} = (1/4)c^2 r_{\rm out}^{-1}$. So

$$f^- = (c^4/16) r_{\rm out}^{-4} f = (16/c^4) r_{\rm in}^4 f, \qquad\qquad \zeta^- = (16/c^4) r_{\rm in}^4 \zeta \tag{56}$$

We shall use spinor identities for the metric

$$\chi = \sigma^2 \zeta \overline{g} + U f d\phi^2 \tag{57}$$

where $U = U(r)$ is a solution of a first order ODE with appropriate boundary conditions to be specified later (see Eq. (113) and Lemma 10.3 below) and

$$\sigma = (X/X_{\rm K})^{1/4} > 0 \tag{58}$$

For $r > M + c$, σ is differentiable and positive. This is because for $r > M + c$, both $X/\sin^2\theta, X_{\rm K}/\sin^2\theta$ are positive and regular on the axis.
We define $\chi^- = \sigma^2 \zeta^- \overline{g} + U f^- d\phi^2$. Then

$$\left(c^4/16\right) r_{\rm in}^{-4} \chi^- = \chi^+ \equiv \chi \tag{59}$$

provided the same function U is used for both the metrics $\chi^\pm$. The actual functions we shall use are not known to be the same initially. However Eq. (59) is useful in transforming formulas.

$r_{\rm out} = r_{\rm in}$ occurs at $r = M \pm c$. At these values $r_{\rm out/in} = \pm c/2 = c/2$ neglecting the negative sign. For a Kerr-Newman solution $\eta_{\rm K}^-$ and $\eta_{\rm K}^+ = \eta_{\rm K}$ match on the boundary sphere of radius $r = M + c = M + \sqrt{M^2 - \mathfrak{e}^2 - \mathfrak{m}^2 - a^2}$ which corresponds to the outer Killing horizon. We have no business inside the outer Killing horizon.

For the general situation let $\eta^+ = \eta$ and let η^- be defined by replacing f and ζ in Eq. (52) with f^- and ζ^-. Asymptotic conditions ensure that η^+ is asymptotically

flat with mass zero and η^- compactifies the infinity. So if we can show that these metric have nonnegative scalar curvature and they match smoothly at the inner boundaries, then positive mass theorem makes them Euclidean. Since we could not directly show that this scalar curvature is nonnegative we follow a detour. Keeping the spinorial proof of the positive mass theorem in mind we construct two spinor identities that solves difficult parts of the problem.

6 Computation in r, θ coordinates

In general we define r, θ coordinates using Eqs (44). Then we get

$$d\rho^2 + dz^2 = \left(r^2 - 2M'r + (M^2 - \mathfrak{e}^2 - \mathfrak{m}^2)\sin^2\theta + a^2\cos^2\theta\right)\left[\frac{dr^2}{r^2 - 2M'r + a^2} + d\theta^2\right] \tag{60}$$

Let $\Pi = \left(r^2 - 2M'r + a^2\right)^{-1} dr^2 + d\theta^2$. We have

$$\begin{aligned} \overline{g} &= \Omega\left(r^2 - 2M'r + (M^2 - \mathfrak{e}^2 - \mathfrak{m}^2)\sin^2\theta + a^2\cos^2\theta\right)\Pi && (61)\\ \overline{g}_{\theta\theta} &= \Omega\left(r^2 - 2M'r + (M^2 - \mathfrak{e}^2 - \mathfrak{m}^2)\sin^2\theta + a^2\cos^2\theta\right) = |\overline{\nabla}\theta|^{-2} && (62)\\ \overline{g}_{rr} &= \overline{g}_{\theta\theta}\left(r^2 - 2M'r + a^2\right)^{-1} = |\overline{\nabla}r|^{-2} && (63) \end{aligned}$$

We note that for $r \geq M + c$, the expression $r^2 - 2M'r + (M^2 - \mathfrak{e}^2 - \mathfrak{m}^2)\sin^2\theta + a^2\cos^2\theta \geq (M^2 - \mathfrak{e}^2 - \mathfrak{m}^2 - a^2)\sin^2\theta$ is positive away from the axis because we are assuming $M^2 > \mathfrak{e}^2 + \mathfrak{m}^2 + a^2$. The expression is also positive on the axis for $r > M + c$. It is useful to remember the formulas

$$\begin{aligned} r^2 - 2M'r + (M^2 - \mathfrak{e}^2 - \mathfrak{m}^2)\sin^2\theta + a^2\cos^2\theta &= (r - M - c)(r - M + c) + c^2\sin^2\theta\\ r^2 - 2M'r + a^2 &= (r - M - c)(r - M + c) \end{aligned}$$

Only nontrivial Christoffel symbol of Π is $\Gamma^r_{\Pi rr} = -\left(r^2 - 2M'r + a^2\right)^{-1}(r - M)$. All other Christoffel symbol of Π vanish. So using $\overline{\Delta}_{s\Pi}u = s^{-1}\overline{\Delta}_{\Pi}u$ and Eq. (61) we get

$$\begin{aligned} \overline{\Delta}r &= (r - M)(r^2 - 2M'r + a^2)^{-1}|\overline{\nabla}r|^2 && (64)\\ \overline{\Delta}\theta &= 0 && (65) \end{aligned}$$

We note that

$$\begin{aligned} \overline{\Delta} \ln\left(f/\sin^2\theta\right) &= \overline{\Delta} \ln r_{\text{out}} = 0 \\ \overline{\Delta} \ln f = \overline{\Delta} \ln(\sin^2\theta) &= -2\overline{g}^{\theta\theta} \csc^2\theta \end{aligned} \tag{66}$$

The first equation follows because in $\mathbb{R}^2$, $\ln r_{\text{out}}$ is a harmonic function. It can also be checked by explicit calculation using Eq. (64). Similarly the second equation follows because by virtue of Eq. (65), $\overline{\Delta} \ln f = -2 \csc^2\theta |\overline{\nabla}\theta|^2$.

Eqs. (53) give $\dfrac{dr_{\text{out/in}}}{dr} = \pm \dfrac{r_{\text{out/in}}}{\sqrt{r^2 - 2M'r + a^2}}$. For a differentiable function $U = U(r)$ for $r > M + c$,

$$\lim_{r \to (M+c)^+} \frac{d \ln U}{dr_{\text{out/in}}} = \pm \lim_{r \to (M+c)^+} \frac{2\sqrt{r^2 - 2M'r + a^2}}{c} \frac{d \ln U}{dr} \tag{67}$$

where $+$ sign of $\pm$ is for r_{out}. These equations need some clarification because finally we shall arrange such that at $r_{\text{out/in}} = c/2$, $\dfrac{d \ln U}{dr_{\text{out}}} = \dfrac{d \ln U}{dr_{\text{in}}}$. Thus in the RHS of Eq. (67), $U = U(r)$ are two different functions $U^\pm$ of r unless $(d \ln U/dr)$ vanishes.

7 Scalar curvature of the 3-metric χ

In the following when we use the symbols $X_{\text{K}}, W_{\text{K}}, V_{\text{K}}, \psi_{\text{K}}, \xi_{\text{K}}$ and Ω_{K} we mean functions defined on Σ^+, $r > M + c$ and these functions have the same functions of the newly defined variables r, θ on Σ^+, $r > M + c$ as those of respective functions in the Kerr-Newman solution in the usual r, θ coordinates of that solution. Once we establish the uniqueness these two sets of functions will be the same object. For example X_{K} has a factor of $\sin^2\theta$ so it will vanish in the limit as $r \downarrow M+c, \sin\theta \downarrow 0$. Now on Σ^+ this limiting set are the finite parts of the axis between two black holes in addition to the topmost and bottommost poles (and possibly some parts of the axis attached to these two poles) but for Kerr-Newman solution this set consists of only two poles. We cannot expect the set $r \downarrow M + c$ to have the same horizon-like prorerty for X_{K} on Σ^+ as in the Kerr-Newman solution unless we can show $\Omega_{\text{K}} = \Omega$ on Σ^+. Also we note that $r = M + c$ set cannot intersect as a curve transversely the black horizon away from the poles because by Eq. (45), $\rho \to 0$ on the $r = M + c$ set and then ρ would be 0 away from the axis and horizon. On

the other hand there is a curve $r = M + c + \epsilon$ for some positive ϵ close to the black hole horizon because the r, θ coordinates are regular there and ϵ must tend to 0 as the horizon is approached with $\sin\theta \neq 0$ on the horizon. Thus all the black holes will be enclosed inside the limiting set $r \downarrow M + c$.

We compute the scalar curvature R_χ of χ defined in Eq (57). For convenience we write

$$f_{\rm em} = X^{-1}|\overline{\nabla}\psi|^2 + X^{-1}\left|X\overline{\nabla}\xi + W\overline{\nabla}\psi\right|^2\rho^{-2} - X_{\rm K}^{-1}|\overline{\nabla}\psi_{\rm K}|^2 - X_{\rm K}^{-1}\left|X_{\rm K}\overline{\nabla}\xi_{\rm K} + W_{\rm K}\overline{\nabla}\psi_{\rm K}\right|^2\rho^{-2}$$

As explained after defining σ in Eq. (58) the expression $\left\langle\overline{\nabla}\ln\left(X_{\rm K}/\sqrt{\rho}\right), \overline{\nabla}\ln\sigma\right\rangle$ is well-defined for $r > M + c$.

Lemma 7.1.

$$\sigma^2\zeta R_\chi = \frac{1}{2}\left|\overline{\nabla}\ln U\right|^2 + \mathcal{P} - U^{-1}\left(\overline{\Delta}U + \left\langle\overline{\nabla}U, \overline{\nabla}\ln f\right\rangle + Q_{\overline{g}}U\right). \tag{68}$$

where $Q_{\overline{g}}$ and $\mathcal{P}$ are as follows.

$$Q_{\overline{g}} = \left(4\left\langle\overline{\nabla}\ln(X_K/\sqrt{\rho}), \overline{\nabla}\ln\sigma\right\rangle + f_{em}\right)^- \tag{69}$$

$$\mathcal{P} = \left(4\left\langle\overline{\nabla}\ln(X_K/\sqrt{\rho}), \overline{\nabla}\ln\sigma\right\rangle + f_{em}\right)^+ + 8|\overline{\nabla}\ln\sigma|^2 \tag{70}$$

Proof. For a given function $\widetilde{f}$ the scalar curvature R_γ of $\gamma = \overline{g} + \widetilde{f}d\phi^2$ is given by

$$R_\gamma = \overline{R} - \widetilde{f}^{-1}\overline{\Delta}\widetilde{f} + \frac{1}{2}|\overline{\nabla}\ln\widetilde{f}|^2 \tag{71}$$

Let $\widetilde{f} = f\zeta^{-1}$. Then $\eta = \zeta\gamma$. So using the conformal transformation formula

$$\eta = \Psi^4\gamma, \quad \Psi^4 R_\eta = R_\gamma - 8\Psi^{-1}\Delta_\gamma\Psi = R_\gamma - 8\Delta_\gamma\ln\Psi - 8|\nabla\ln\Psi|^2_\gamma \tag{72}$$

and writing the Laplacian Δ_γ relative to the 3-metric γ in terms of the Laplacian of $\bar{g}$ using

$$\Delta_\gamma u = \overline{\Delta}u + (1/2)\left\langle\overline{\nabla}\ln\widetilde{f}, \overline{\nabla}u\right\rangle \tag{73}$$

we get $\zeta R_\eta = R_\gamma - 2\zeta^{-1}\overline{\Delta}\zeta + (3/2)\zeta^{-2}|\overline{\nabla}\zeta|^2 - \left\langle\overline{\nabla}\ln\widetilde{f}, \overline{\nabla}\ln\zeta\right\rangle$. Using Eqs. (71,37) we then get

$$\begin{aligned}\zeta R_\eta = {} & (1/2)X^{-1}\overline{\Delta}X - (1/2)\left\langle\overline{\nabla}\ln\rho, \overline{\nabla}\ln X\right\rangle + X^{-1}|\overline{\nabla}\psi|^2 + X^{-1}\left|X\overline{\nabla}\xi + W\overline{\nabla}\psi\right|^2\rho^{-2} \\ & - \widetilde{f}^{-1}\overline{\Delta}\widetilde{f} + (1/2)|\overline{\nabla}\ln\widetilde{f}|^2 - 2\zeta^{-1}\overline{\Delta}\zeta + (3/2)|\overline{\nabla}\ln\zeta|^2 - \left\langle\overline{\nabla}\ln\widetilde{f}, \overline{\nabla}\ln\zeta\right\rangle\end{aligned} \tag{74}$$

Since for Kerr-Newman this gives

$$
\begin{aligned}
0 = (1/2)X_{\mathrm{K}}^{-1}\overline{\Delta}X_{\mathrm{K}} - (1/2)\left\langle\overline{\nabla}\ln\rho, \overline{\nabla}\ln X_{\mathrm{K}}\right\rangle + X_{\mathrm{K}}^{-1}|\overline{\nabla}\psi_{\mathrm{K}}|^2 + X_{\mathrm{K}}^{-1}\left|X_{\mathrm{K}}\overline{\nabla}\xi_{\mathrm{K}} + W_{\mathrm{K}}\overline{\nabla}\psi_{\mathrm{K}}\right|^2\rho^{-2} \\
- \widetilde{f}^{-1}\overline{\Delta}\widetilde{f} + (1/2)|\overline{\nabla}\ln\widetilde{f}|^2 - 2\zeta^{-1}\overline{\Delta}\zeta + (3/2)|\overline{\nabla}\ln\zeta|^2 - \left\langle\overline{\nabla}\ln\widetilde{f}, \overline{\nabla}\ln\zeta\right\rangle
\end{aligned}
$$

So using f_{em} we get

$$
\begin{aligned}
\zeta R_\eta &= (1/2)X^{-1}\overline{\Delta}X - (1/2)\left\langle\overline{\nabla}\ln\rho, \overline{\nabla}\ln X\right\rangle + f_{\mathrm{em}} - (1/2)X_{\mathrm{K}}^{-1}\overline{\Delta}X_{\mathrm{K}} + (1/2)\left\langle\overline{\nabla}\ln\rho, \overline{\nabla}\ln X_{\mathrm{K}}\right\rangle \\
&= (1/2)\overline{\Delta}\ln X + (1/2)|\overline{\nabla}\ln X|^2 - (1/2)\overline{\Delta}\ln X_{\mathrm{K}} - (1/2)|\overline{\nabla}\ln X_{\mathrm{K}}|^2 - (1/2)\left\langle\overline{\nabla}\ln\rho, \overline{\nabla}\ln(X/X_{\mathrm{K}})\right\rangle + f_{\mathrm{em}} \\
&= \frac{1}{2}\overline{\Delta}\ln(X/X_{\mathrm{K}}) + (1/2)\left\langle\overline{\nabla}\ln(XX_{\mathrm{K}}/\rho), \overline{\nabla}\ln(X/X_{\mathrm{K}})\right\rangle + f_{\mathrm{em}} \\
&= (1/2)\overline{\Delta}\ln(X/X_{\mathrm{K}}) + (1/2)|\overline{\nabla}\ln(X/X_{\mathrm{K}})|^2 + (1/2)\left\langle\overline{\nabla}\ln(X_{\mathrm{K}}^2/\rho), \overline{\nabla}\ln(X/X_{\mathrm{K}})\right\rangle + f_{\mathrm{em}} \\
&= 2\overline{\Delta}\ln\sigma + 8|\overline{\nabla}\ln\sigma|^2 + 2\left\langle\overline{\nabla}\ln(X_{\mathrm{K}}^2/\rho), \overline{\nabla}\ln\sigma\right\rangle + f_{\mathrm{em}}
\end{aligned}
$$

where in the last step we used Eq. (58). Remembering $Q_{\overline{g}}$ and $\mathcal{P}$ we get

$$\zeta R_\eta = 2\overline{\Delta}\ln\sigma + \mathcal{P} - Q_{\overline{g}} \tag{75}$$

Next we compute the scalar curvature of

$$\vartheta = \sigma^2\eta \tag{76}$$

$R_\vartheta = \sigma^{-2}R_\eta - 4\sigma^{-3}\Delta_\eta\sigma + 2\sigma^{-4}|\nabla\sigma|_\eta^2 = \sigma^{-2}R_\eta - 4\sigma^{-2}\Delta_\eta\ln\sigma - 2\sigma^{-2}|\nabla\ln\sigma|_\eta^2$. Using Eq. (52) and the formula Eq. (73) we have

$$\Delta_\eta\ln\sigma = \overline{\Delta}_{\zeta\overline{g}}\ln\sigma + \frac{1}{2f}\left\langle\overline{\nabla}f, \overline{\nabla}\ln\sigma\right\rangle_{\zeta\overline{g}}$$

Thus $\zeta\sigma^2 R_\vartheta = \zeta R_\eta - 4\overline{\Delta}\ln\sigma - 2\left\langle\overline{\nabla}\ln f, \overline{\nabla}\ln\sigma\right\rangle - 2|\nabla\ln\sigma|^2$. Using Eq. (75) we get

$$\zeta\sigma^2 R_\vartheta = -2\overline{\Delta}\ln\sigma + \mathcal{P} - Q_{\overline{g}} - 2\left\langle\overline{\nabla}\ln\sigma, \overline{\nabla}\ln f\right\rangle - 2|\overline{\nabla}\ln\sigma|^2 \tag{77}$$

Finally we write χ as $\vartheta + \varpi d\phi^2$. $\chi_{\phi\phi} = Uf = \sigma^2 f + \varpi \Rightarrow U = f^{-1}(\sigma^2 f + \varpi)$. We recall that if $h = \overline{G} + \varphi d\phi^2$, and $\widehat{h} = \overline{G} + \widehat{\varphi}d\phi^2$, where $\overline{G}$ is a 2-dimensional metric on the ϕ =constant surfaces and $\varphi, \widehat{\varphi}$ are independent of ϕ, then

$$R_h = R_{\widehat{h}} + \overline{\Delta}_{\overline{G}}\ln\frac{\widehat{\varphi}}{\varphi} - \frac{1}{2}|\overline{\nabla}\ln\varphi|_{\overline{G}}^2 + \frac{1}{2}|\overline{\nabla}\ln\widehat{\varphi}|_{\overline{G}}^2 \tag{78}$$

Taking $\overline{G} = \sigma^2\zeta\overline{g}$, $\varphi = \sigma^2 f + \varpi = \chi_{\phi\phi}$and $\widehat{\varphi} = \sigma^2 f = \vartheta_{\phi\phi}$ we get

$$\begin{aligned}
\sigma^2\zeta R_\chi &= \sigma^2\zeta\left(R_\vartheta + \overline{\Delta}_{\sigma^2\zeta\overline{g}} \ln \frac{\sigma^2 f}{\sigma^2 f + \varpi} - (1/2)|\overline{\nabla} \ln\left(\sigma^2 f + \varpi\right)|^2_{\overline{G}} + (1/2)|\overline{\nabla} \ln\left(\sigma^2 f\right)|^2_{\overline{G}}\right) \\
&= \sigma^2\zeta\left(R_\vartheta + \sigma^{-2}\zeta^{-1}\overline{\Delta} \ln \frac{\sigma^2 f}{\sigma^2 f + \varpi} - (1/2)\sigma^{-2}\zeta^{-1}|\overline{\nabla} \ln\left(\sigma^2 f + \varpi\right)|^2 + (1/2)\sigma^{-2}\zeta^{-1}|\overline{\nabla} \ln\left(\sigma^2 f\right)|^2\right) \\
&= \sigma^2\zeta R_\vartheta + \overline{\Delta} \ln(\sigma^2 U^{-1}) - (1/2)|\overline{\nabla} \ln(Uf)|^2 + (1/2)|\overline{\nabla} \ln\left(\sigma^2 f\right)|^2 \\
&= \sigma^2\zeta R_\vartheta + \overline{\Delta} \ln(\sigma^2 U^{-1}) - (1/2)|\overline{\nabla} \ln U|^2 + 2|\overline{\nabla} \ln \sigma|^2 - \left\langle \overline{\nabla} \ln U, \overline{\nabla} \ln f\right\rangle + 2\left\langle \overline{\nabla} \ln \sigma, \overline{\nabla} \ln f\right\rangle \\
&= -2\overline{\Delta} \ln \sigma + \mathcal{P} - Q_{\overline{g}} + \overline{\Delta} \ln(\sigma^2 U^{-1}) - (1/2)|\overline{\nabla} \ln U|^2 - \left\langle \overline{\nabla} \ln U, \overline{\nabla} \ln f\right\rangle
\end{aligned}$$

where in the last step we used Eq. (77). Thus we get Eq. (68). □

Similarly we find the scalar curvature of $\chi^- = \sigma^2\zeta^-\overline{g} + Uf^- d\phi^2$ to be

$$R_{\chi^-} = (\zeta^-)^{-1}\sigma^{-2}\left(\mathcal{P} - Q_{\overline{g}} - U^{-1}\overline{\Delta} U + \frac{1}{2}|\overline{\nabla} \ln U|^2 - \left\langle \overline{\nabla} \ln U, \overline{\nabla} \ln f^-\right\rangle\right) \tag{79}$$

We can also derive this formula by conformal transformation $\chi^- = (16/c^4)r_{\text{in}}^4\chi$ (see Eq. (59)) and the fact that $\ln r_{\text{in}}$ is a harmonic function in the 2-metric $\overline{g}$.

8 Finding a Spinor

Let r_0 be a constant. Let $\oiint\limits_{r=r_0,\eta}$ represents the surface integral on the $r = r_0$ surface relative to the 2-metric induced from η and $\iint\limits_{r=r_0}$ represents ordinary double integral. All integrations are done on subsets of $\Sigma^+ \cup \partial\Sigma^+$ unless indicated otherwise.

We need a $SU(2)$-spinor Θ_ϑ on Σ with the following properties.

$$D_\vartheta\Theta_\vartheta = 0 \tag{80}$$

$$||\Theta_\vartheta|| = 1 + O(r^{-1}) \text{ as } r \to \infty \tag{81}$$

$$\frac{\partial||\Theta_\vartheta||^2}{\partial r} = o(r^{-1}) \text{ as } r \to \infty \tag{82}$$

$$\Theta_\vartheta \text{ is independent of } \phi \tag{83}$$

$$\text{As } r_0 \downarrow M + c \text{ on the horizon } \int\limits_{r=r_0} \sqrt{r^2 - 2M'r + a^2}\frac{\partial||\Theta_\vartheta||^2}{\partial r} d\theta = 0 \tag{84}$$

Here D_ϑ is the Dirac operator of the metric ϑ. Such a spinor exists. On the double the 3-metric $\widehat{g}$ has nonnegative scalar curvature, and it is asymptotically flat. So we can use Bartnik's proof for the existence of a spinor $\Theta_{\widehat{g}}$ harmonic relative to $\widehat{g}$. Because of the axisymmtery we can choose $\Theta_{\widehat{g}}$ to be independent of ϕ. Θ_ϑ can be obtained $\Theta_{\widehat{g}}$ by what we called a 2+1 conformal transformation. It is explained below. We have outlined the proof of the following lemma in the appendix.

Lemma 8.1. *Let $\overline{G} = \overline{G}_{11}(d(x^1)^2 + d(x^2)^2)$, $g_1 = \overline{G} + f_1 d\phi^2$ and $g_2 = \overline{G} + qf_1 d\phi^2$. All functions and metrics are independent of ϕ. If Θ is a spinor satisfying the Dirac equation $D_{g_1}\Theta = 0$ and Θ is independent of ϕ, then*

$$D_{g_2}\left(q^{-\frac{3}{8}}\Theta\right) = 0$$

We also have the conformal transformation formula. Let $\xi_{\psi^{-2}\widehat{\chi}}$ be a fixed spinor satisfying Dirac equation relative to the metric $\psi^{-2}\widehat{\chi}$. Then the spinor $\xi_{\widehat{\chi}} = \psi^{-1}\xi_{\psi^{-2}\widehat{\chi}}$ satisfies Dirac equation relative to the conformal metric $\widehat{\chi}$ (Lichnerowicz [19], Branson, T., Kosmann-Schwarzbach [20]).

To find the spinor Θ_ϑ from $\Theta_{\widehat{g}}$ we take $g_1 = \widehat{g} = \overline{g} + Xd\phi^2$ and $g_2 = \sigma^{-2}\zeta^{-1}\vartheta = \overline{g} + \zeta^{-1}fd\phi^2$ in Lemma 8.1. That is we put $f_1 = X$ and $q = f\zeta^{-1}X^{-1}$. Then $D_{\sigma^{-2}\zeta^{-1}\vartheta}\left(\left(f\zeta^{-1}X^{-1}\right)^{-3/8}\Theta_{\widehat{g}}\right) = 0$. So the spinor

$$\Theta_\vartheta = \sigma^{-1}\zeta^{-1/2}\left(f\zeta^{-1}X^{-1}\right)^{-3/8}\Theta_{\widehat{g}} \tag{85}$$

satisfies $D_\vartheta\Theta_\vartheta = 0$. We note that

$$\|\Theta_\vartheta\|^2 = \sigma^{-2}\zeta^{-1/4}X^{3/4}f^{-3/4}\|\Theta_{\widehat{g}}\|^2 \tag{86}$$

Now $8\pi M = -\oiint_{\lambda_0\to\infty,\overline{g}}\left\langle\nabla_{\widehat{g}}\|\Theta_{\widehat{g}}\|^2, n_{\widehat{g}}\right\rangle_{\widehat{g}}$ so that (because of the asymptotic regularity in the existence proof of the spinor)

$$\|\Theta_{\widehat{g}}\|^2 = 1 - 2M/r + O(r^{-2}) \tag{87}$$

which by virtue of the asymptotic conditions on σ, f, ζ, X gives

$$\|\Theta_\vartheta\|^2 = 1 + O(r^{-2}) \tag{88}$$

Thus Θ_ϑ satisfies properties 80-83. To see Eq. (84) we note that $\int\limits_{r=r_0} \sqrt{r^2 - 2M'r + a^2} \frac{\partial \|\Theta_\vartheta\|^2}{\partial r} d\theta = - \oint\limits_{r_0, \overline{\vartheta}} \left\langle \overline{\nabla} \|\Theta_\vartheta\|^2, n_{\overline{\vartheta}} \right\rangle_{\overline{\vartheta}}$ where $n_{\overline{\vartheta}}$ is the unit normal form on the r = constant loop on a ϕ = constant surface with the normal vector pointing towards decreasing r. $\Theta_{\widehat{g}}$ is a harmonic spinor on the double $\Sigma^+ \cup \Sigma^- \cup \partial\Sigma^+$ relative to the metric $\widehat{g}$ producing the same contribution at each end for $\lim\limits_{r_0 \uparrow \infty} \oiint\limits_{r_0, \widehat{g}} \left\langle \overline{\nabla} \|\Theta_{\widehat{g}}\|^2, n_{\widehat{g}} \right\rangle_{\widehat{g}}$. Thus on the horizon $\oiint\limits_{\widehat{g}} \left\langle \overline{\nabla} \|\Theta_{\widehat{g}}\|^2, n_{\widehat{g}} \right\rangle_{\widehat{g}} = 0$ because of the symmetry across the totally geodesic boundary where $n_{\widehat{g}}$ is the unit normal form on the relevant surface. Since the derivative of $\sigma^{-2}\zeta^{-1/4}X^{3/4}f^{-3/4}$ is regular on the horizon we get Eq. (84).

Now we find a spinor satisfying $D_\chi \Theta_\chi = 0$. Recalling Eqs. (76,52,57) for χ and ϑ we apply Lemma 8.1 again with $g_1 = \vartheta$, $f_1 = f\sigma^2$, $g_2 = \chi$ and $q = U\sigma^{-2}$. Thus we take

$$\Theta_\chi = \sigma^{\frac{3}{4}} U^{-\frac{3}{8}} \Theta_\vartheta \tag{89}$$

For the harmonic spinor Θ_χ one has the identity,

$$2\Delta_\chi \|\Theta_\chi\|^2 = R_\chi \|\Theta_\chi\|^2 + 4\|\nabla_\chi \Theta_\chi\|^2 \tag{90}$$

Using the expression for R_χ from Eq. (68) we get

$$2\Delta_\chi \|\Theta_\chi\|^2 = \left((1/2)\left\langle \overline{\nabla} \ln U, \overline{\nabla} \ln U \right\rangle + \mathcal{P} - U^{-1}\left(\overline{\Delta} U + \left\langle \overline{\nabla} U, \overline{\nabla} \ln f \right\rangle + Q_{\overline{g}} U\right)\right) \sigma^{-2}\zeta^{-1} \|\Theta_\chi\|^2 + 4\|\nabla_\chi \Theta_\chi\|^2 \tag{91}$$

For complex U, $|\overline{\nabla} \ln U|^2$ is not nonnegative definite. So we have replaced it in Eq. (68) by $\left\langle \overline{\nabla} \ln U, \overline{\nabla} \ln U \right\rangle$ to remove confusion. Our next aim is to write the above identity using 2-dimensional Laplacian because that way we can easily tackle integration if U becomes complex.

9 Two spinor identities on $\Sigma^\pm$

Let $L = (r, \theta)$

$$|U|^{-\frac{3}{4}} / \|\Theta_\chi\|^2 = L \tag{92}$$

L is not defined on the possible zero set of $\|\Theta_\chi\|$ in case the known spinor $\Theta_{\widehat{g}}$ can vanish. By Eq. (89), $\sigma^{3/2} L \|\Theta_\vartheta\|^2 = 1$. We need to introduce L on $\Sigma^\pm$ only for

$L_{ave} \leq 4/3$. This will be clear later from Eqs. (115,120). $L_{\text{ave}} = L_{\text{ave}}(r), r > M + c$ is defined as follows.

$$L_{ave}(r_0) \int\limits_{r=r_0} \sigma^{3/2} ||\Theta_\vartheta||^2 \sin^2\theta d\theta = \int\limits_{r=r_0} \sigma^{3/2} L ||\Theta_\vartheta||^2 \sin^2\theta d\theta = \int\limits_{r=r_0} \sin^2\theta d\theta \quad (93)$$

We shall use L either in the form $L||\Theta_\vartheta||^2$ or in L_{ave}. We removed the θ dependence from $L = L(r, \theta)$ by averaging L on the r = constant loops on a ϕ = constant surface. For future reference we note that L and $L_{\text{ave}} \to 1$ as $r \to \infty$. This follows from Eq. (81). We have introduced L to present some equations in a compact form. We can possibly take the mystery out of L by examining the following cumbersome expression for L.

$$L = X_{\text{K}}^{-1/8} \zeta^{1/4} X^{-5/8} f^{3/4} ||\Theta_{\widehat{g}}||^{-2}$$

Thus as $\sin\theta \to 0$ but $r > M + c$, $L = O(1)\zeta^{1/4}||\Theta_{\widehat{g}}||^{-2}$. As $\sin\theta \to 0$ and $r \downarrow M + c$, $L = O((\sin\theta)^{1/4})\zeta^{1/4}||\Theta_{\widehat{g}}||^{-2}$. Thus L tends to a finite limit as $r \downarrow M + c$ on the horizon whenever the spinor $\Theta_{\widehat{g}} \neq 0$ on the horizon. Since $\Theta_{\widehat{g}}$ cannot identically vanish on the totally geodesic surface representing the horizon, Eq. (93) shows that L_{ave} is bounded as $r \downarrow M + c$. However as stated above we are interested only for $L_{ave} \leq 4/3$.

We denote a ϕ = constant surface in Σ^+ by Σ_2^+. Let $U = |U|e^{i\omega}$, ω being real. We assume that U, ω are functions of r only.

Lemma 9.1. *On $(\Sigma_2^+, \overline{\chi})$ for $U = U(r)$ wherever $U = U(r)$ is twice differentiable,*

$$\overline{\nabla}_{\overline{\chi}} \left(\sin^2\theta \left(2\overline{\nabla}||\Theta_\chi||^2 + 2||\Theta_\chi||^2 \overline{\nabla} \ln r_{out} - 2|U|^{-\frac{3}{4}} \overline{\nabla} \ln r_{out} + ||\Theta_\chi||^2 \overline{\nabla} \ln U \right) - (||\Theta_\chi||^2 - 1)\overline{\nabla} \sin^2\theta \right) =$$
$$- ||\Theta_\chi||^2 \overline{\Delta}_{\overline{\chi}} \sin^2\theta + \overline{\Delta}_{\overline{\chi}} \sin^2\theta + 4\sin^2\theta ||\nabla_\chi \Theta_\chi||^2 - \frac{3}{4} iL \left\langle \overline{\nabla}\omega, \overline{\nabla} \ln f \right\rangle_{\overline{\chi}} ||\Theta_\chi||^2 \sin^2\theta +$$
$$\sin^2\theta \left(\mathcal{P} - \frac{1}{2} \left\langle \overline{\nabla} \ln U, \overline{\nabla} \ln U \right\rangle - \frac{1}{4}(4 - 3L) \left\langle \overline{\nabla} \ln U, \overline{\nabla} \ln f \right\rangle - Q_{\overline{g}} \right) \zeta^{-1} ||\Theta_\chi||^2 \quad (94)$$

Proof. Writing the 3-Laplacian Δ_χ relative to $\chi = \sigma^2\zeta\overline{g} + U f d\phi^2$ in terms of the Laplacian of the 2-metric $\overline{\chi} = \sigma^2\zeta\overline{g}$ we get $2\Delta_\chi||\Theta_\chi||^2 = 2\overline{\Delta}_{\overline{\chi}}||\Theta_\chi||^2 + \left\langle \overline{\nabla} \ln (U f), \overline{\nabla}||\Theta_\chi||^2 \right\rangle_{\overline{\chi}}$. We also have $U^{-1}||\Theta_\chi||^2 \overline{\Delta}_{\overline{\chi}} U = \overline{\nabla}_{\overline{\chi}} \left(||\Theta_\chi||^2 U^{-1} \overline{\nabla} U \right) - \left\langle \overline{\nabla}||\Theta_\chi||^2, \overline{\nabla} \ln U \right\rangle_{\overline{\chi}} + \left\langle \overline{\nabla} \ln U, \overline{\nabla} \ln U \right\rangle_{\overline{\chi}} ||\Theta_\chi||^2$. So Eq. (91) gives $2\overline{\Delta}_{\overline{\chi}}||\Theta_\chi||^2 + \left\langle \overline{\nabla} \ln f, \overline{\nabla}||\Theta_\chi||^2 \right\rangle_{\overline{\chi}} + \overline{\nabla}_{\overline{\chi}} \left(||\Theta_\chi||^2 U^{-1} \overline{\nabla} U \right) =$
$\left(\mathcal{P} - (1/2) \left\langle \overline{\nabla} \ln U, \overline{\nabla} \ln U \right\rangle - \left\langle \overline{\nabla} \ln U, \overline{\nabla} \ln f \right\rangle - Q_{\overline{g}} \right) \sigma^{-2} \zeta^{-1} ||\Theta_\chi||^2 + 4||\nabla_\chi \Theta_\chi||^2 \Rightarrow$
$2\overline{\Delta}_{\overline{\chi}}||\Theta_\chi||^2 + \overline{\nabla}_{\overline{\chi}} \left(||\Theta_\chi||^2 \overline{\nabla} \ln f - 2|U|^{-3/4} \overline{\nabla} \ln r_{\text{out}} + ||\Theta_\chi||^2 \overline{\nabla} \ln U \right) = 4||\nabla_\chi \Theta_\chi||^2 + ||\Theta_\chi||^2 \sigma^{-2} \zeta^{-1} \overline{\Delta} \ln f +$

$(3/2)|U|^{-3/4}\left\langle\overline{\nabla}\ln|U|,\overline{\nabla}\ln r_{\text{out}}\right\rangle+\left(\mathcal{P}-(1/2)\left\langle\overline{\nabla}\ln U,\overline{\nabla}\ln U\right\rangle-\left\langle\overline{\nabla}\ln U,\overline{\nabla}\ln f\right\rangle-Q_{\overline{g}}\right)\sigma^{-2}\zeta^{-1}||\Theta_\chi||^2$. Now we use Eq. (66) to get

$$\overline{\nabla}_{\overline{\chi}}\left(2\overline{\nabla}||\Theta_\chi||^2+||\Theta_\chi||^2\overline{\nabla}\ln f-2|U|^{-3/4}\overline{\nabla}\ln r_{\text{out}}+||\Theta_\chi||^2\overline{\nabla}\ln U\right)=4||\nabla_\chi\Theta_\chi||^2-2||\Theta_\chi||^2\sigma^{-2}\zeta^{-1}\overline{g}^{\theta\theta}\csc^2\theta+$$
$$(3/2)|U|^{-3/4}\left\langle\overline{\nabla}\ln|U|,\overline{\nabla}\ln r_{\text{out}}\right\rangle_{\overline{\chi}}+\left(\mathcal{P}-(1/2)\left\langle\overline{\nabla}\ln U,\overline{\nabla}\ln U\right\rangle-\left\langle\overline{\nabla}\ln U,\overline{\nabla}\ln f\right\rangle-Q_{\overline{g}}\right)\sigma^{-2}\zeta^{-1}||\Theta_\chi||^2 \quad (95)$$

For $U=U(r)$, $2\left\langle\overline{\nabla}\ln|U|,\overline{\nabla}\ln r_{\text{out}}\right\rangle=\left\langle\overline{\nabla}\ln|U|,\overline{\nabla}\ln f\right\rangle=\left\langle\overline{\nabla}\ln U,\overline{\nabla}\ln f\right\rangle-i\left\langle\overline{\nabla}\omega,\overline{\nabla}\ln f\right\rangle$. So RHS of Eq. (95) simplifies to

$$4||\nabla_\chi\Theta_\chi||^2-2||\Theta_\chi||^2\sigma^{-2}\zeta^{-1}\overline{g}^{\theta\theta}\csc^2\theta-(3/4)iL\left\langle\overline{\nabla}\omega,\overline{\nabla}\ln f\right\rangle_{\overline{\chi}}||\Theta_\chi||^2+$$
$$\left(\mathcal{P}-(1/2)\left\langle\overline{\nabla}\ln U,\overline{\nabla}\ln U\right\rangle-\frac{1}{4}(4-3L)\left\langle\overline{\nabla}\ln U,\overline{\nabla}\ln f\right\rangle-Q_{\overline{g}}\right)\sigma^{-2}\zeta^{-1}||\Theta_\chi||^2$$

Now we multiply both sides of Eq. (95) by $\sin^2\theta$. LHS becomes (since U, r_{out} are functions of r only),

$$\overline{\nabla}_{\overline{\chi}}\left(\sin^2\theta\left(2\overline{\nabla}||\Theta_\chi||^2+||\Theta_\chi||^2\overline{\nabla}\ln f-2|U|^{-3/4}\overline{\nabla}\ln r_{\text{out}}+||\Theta_\chi||^2\overline{\nabla}\ln U\right)\right)-$$
$$2\left\langle\overline{\nabla}\sin^2\theta,\overline{\nabla}||\Theta_\chi||^2\right\rangle_{\overline{\chi}}-\left\langle\overline{\nabla}\sin^2\theta,\overline{\nabla}\ln(\sin^2\theta)\right\rangle_{\overline{\chi}}||\Theta_\chi||^2=$$
$$\overline{\nabla}_{\overline{\chi}}\left(\sin^2\theta\left(2\overline{\nabla}||\Theta_\chi||^2+||\Theta_\chi||^2\overline{\nabla}\ln f-2|U|^{-3/4}\overline{\nabla}\ln r_{\text{out}}+||\Theta_\chi||^2\overline{\nabla}\ln U\right)\right)-2\overline{\nabla}_{\overline{\chi}}\left(||\Theta_\chi||^2\overline{\nabla}_{\overline{\chi}}\sin^2\theta\right)+2||\Theta_\chi||^2\overline{\Delta}_{\overline{\chi}}\sin^2\theta$$
$$-4\sigma^{-2}\zeta^{-1}\overline{g}^{\theta\theta}||\Theta_\chi||^2\cos^2\theta=$$
$$\overline{\nabla}_{\overline{\chi}}\left(\sin^2\theta\left(2\overline{\nabla}||\Theta_\chi||^2+2||\Theta_\chi||^2\overline{\nabla}\ln r_{\text{out}}+2||\Theta_\chi||^2\overline{\nabla}\ln\sin\theta-2|U|^{-3/4}\overline{\nabla}\ln r_{\text{out}}+||\Theta_\chi||^2\overline{\nabla}\ln U\right)\right)-2\overline{\nabla}_{\overline{\chi}}\left(||\Theta_\chi||^2\overline{\nabla}_{\overline{\chi}}\sin^2\theta\right)$$
$$+2||\Theta_\chi||^2\overline{\Delta}_{\overline{\chi}}\sin^2\theta-4\sigma^{-2}\zeta^{-1}\overline{g}^{\theta\theta}||\Theta_\chi||^2\cos^2\theta=$$
$$\overline{\nabla}_{\overline{\chi}}\left(\sin^2\theta\left(2\overline{\nabla}||\Theta_\chi||^2+2||\Theta_\chi||^2\overline{\nabla}\ln r_{\text{out}}-2|U|^{-3/4}\overline{\nabla}\ln r_{\text{out}}+||\Theta_\chi||^2\overline{\nabla}\ln U\right)-||\Theta_\chi||^2\overline{\nabla}\sin^2\theta\right)+2||\Theta_\chi||^2\overline{\Delta}_{\overline{\chi}}\sin^2\theta$$
$$-4\sigma^{-2}\zeta^{-1}\overline{g}^{\theta\theta}||\Theta_\chi||^2\cos^2\theta=$$
$$\overline{\nabla}_{\overline{\chi}}\left(\sin^2\theta\left(2\overline{\nabla}||\Theta_\chi||^2+2||\Theta_\chi||^2\overline{\nabla}\ln r_{\text{out}}-2|U|^{-3/4}\overline{\nabla}\ln r_{\text{out}}+||\Theta_\chi||^2\overline{\nabla}\ln U\right)-||\Theta_\chi||^2\overline{\nabla}\sin^2\theta\right)-4\sigma^{-2}\zeta^{-1}\overline{g}^{\theta\theta}||\Theta_\chi||^2\sin^2\theta$$

where we used

$$\overline{\Delta}\sin^2\theta=2\overline{g}^{\theta\theta}\cos(2\theta) \quad (96)$$

Thus Eq. (95) becomes

$$\overline{\nabla}_{\overline{\chi}}\left(\sin^2\theta\left(2\overline{\nabla}||\Theta_\chi||^2+2||\Theta_\chi||^2\overline{\nabla}\ln r_{\text{out}}-2|U|^{-\frac{3}{4}}\overline{\nabla}\ln r_{\text{out}}+||\Theta_\chi||^2\overline{\nabla}\ln U\right)-||\Theta_\chi||^2\overline{\nabla}\sin^2\theta\right)=$$
$$-2\sigma^{-2}\zeta^{-1}\overline{g}^{\theta\theta}||\Theta_\chi||^2\cos(2\theta)+4\sin^2\theta||\nabla_\chi\Theta_\chi||^2-(3/4)iL\left\langle\overline{\nabla}\omega,\overline{\nabla}\ln f\right\rangle_{\overline{\chi}}||\Theta_\chi||^2\sin^2\theta$$
$$+\sin^2\theta\left(\mathcal{P}-\frac{1}{2}\left\langle\overline{\nabla}\ln U,\overline{\nabla}\ln U\right\rangle-(1/4)(4-3L)\left\langle\overline{\nabla}\ln U,\overline{\nabla}\ln f\right\rangle-Q_{\overline{g}}\right)\sigma^{-2}\zeta^{-1}||\Theta_\chi||^2 \quad (97)$$

Hence using Eq. (96) again and adding $\overline{\Delta}_{\overline{\chi}}\sin^2\theta$ to both sides we get Eq. (94). □

On $(\Sigma^-, \overline{\chi^-})$ we get a similar identity as Eq. (94) using the spinor $\Theta_{\chi^-} = \sigma^{3/4}U^{-3/8}\Theta_{\vartheta^-}$ (compare Eq. (89)) where $\vartheta^- = \sigma^2\eta^- = \sigma^2\zeta^-\overline{g} + \sigma^2 f^- d\phi^2$ and Θ_{ϑ^-} we define replacing ζ, f by ζ^-, f^- in Eq. (85). Then by Eq. (56), $\Theta_{\chi^-} = (c^2/4)r_{\text{in}}^{-2}\sigma^{3/4}U^{-3/8}\Theta_\vartheta = (c^2/4)r_{\text{in}}^{-2}\Theta_\chi$. This is Θ_χ on Σ^- and it may differ from Θ_χ on Σ^+ because in general $U(r)$ are different functions on $\Sigma^\pm$. However to keep notation simple we use U for $U^\pm$. We also recall $\overline{\chi^-} = (16/c^4)r_{\text{in}}^4\overline{\chi}$. The singular factor r_{in}^{-2} in Θ_{χ^-} makes it difficult to manipulate. So we shall write the identity for Σ^- using Θ_χ on Σ^-. This is done below using several transformation formulas which are straightforward to check.

$$\|\nabla_{\chi^-}(r_{\text{in}}^{-2}\Theta_\chi)\|^2 = 4r_{\text{in}}^{-4}|\nabla \ln r_{\text{in}}|^2_{\chi^-}\|\Theta_\chi\|^2 + r_{\text{in}}^{-4}\|\nabla_{\chi-}\Theta_\chi\|^2 - 2r_{\text{in}}^{-4}\left\langle \nabla \ln r_{\text{in}}, \nabla\|\Theta_\chi\|^2\right\rangle_{\chi^-} \tag{98}$$

$$(16/c^4)\|\nabla_{\chi-}\Theta_\chi\|^2 = r_{\text{in}}^{-4}\|\nabla_\chi\Theta_\chi\|^2 - r_{\text{in}}^{-4}\left\langle \nabla \ln r_{\text{in}}, \nabla\|\Theta_\chi\|^2\right\rangle_\chi + 2r_{\text{in}}^{-4}|\nabla \ln r_{\text{in}}|^2_\chi\|\Theta_\chi\|^2 \tag{99}$$

$$\Delta_{\chi^-}\ln r_{\text{in}} = (1/2)\left\langle \overline{\nabla}\ln(Uf^-), \overline{\nabla}\ln r_{\text{in}}\right\rangle_{\chi^-} \tag{100}$$

As usual when not specified explicitly norms and inner products for gradients of functions are w.r.t. $\overline{g}$. In order to tackle the contribution from the term $\|\Theta_\chi\|^2\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln r_{\text{in}}\right\rangle_{\chi^-}$ coming from Eq. (100) we shall need the following lemma. The lemma is proved in the appendix.

Lemma 9.2. *For the metric $\gamma = \sigma^2\zeta\overline{g} + |U|^{-4}Ufd\phi^2$ and spinor $\Theta_\gamma = |U|^{1/2}\Theta_\chi$,*

$$\|\nabla_\chi\Theta_\chi\|^2 - (1/2)\sigma^{-2}\zeta^{-1}\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln f\right\rangle\|\Theta_\chi\|^2 = iI + |U|^{-1}\|\nabla_\gamma\Theta_\gamma\|^2 \tag{101}$$

where I is a real function. In case U is a positive real function $I = 0$.

The spinor Θ_{χ^-} satisfies $D_{\chi^-}\Theta_{\chi^-} = 0$ by the conformal transformation formula. For it Eq. (90) becomes after dividing out by $c^4/16$,

$$2\Delta_{\chi^-}\|r_{\text{in}}^{-2}\Theta_\chi\|^2 = R_{\chi^-}\|r_{\text{in}}^{-2}\Theta_\chi\|^2 + 4\|\nabla_{\chi^-}(r_{\text{in}}^{-2}\Theta_\chi)\|^2 \tag{102}$$

Using Eqs. (98-100) we obtain

$$\begin{aligned}2\Delta_{\chi^-}\|r_{\text{in}}^{-2}\Theta_\chi\|^2 - 4\|\nabla_{\chi^-}(r_{\text{in}}^{-2}\Theta_\chi)\|^2 &= -4r_{\text{in}}^{-4}\|\Theta_\chi\|^2\left\langle\overline{\nabla}\ln U^-, \overline{\nabla}\ln r_{\text{in}}\right\rangle_{\chi^-}\\ &+ 2r_{\text{in}}^{-4}\Delta_{\chi^-}\|\Theta_\chi\|^2 - 4(c^4/16)r_{\text{in}}^{-8}\|\nabla_\chi\Theta_\chi\|^2 - 4r_{\text{in}}^{-4}\left\langle\nabla\ln r_{\text{in}}, \nabla\|\Theta_\chi\|^2\right\rangle_{\chi^-}\end{aligned} \tag{103}$$

So Eq. (102) gives

$$2\Delta_{\chi^-}\|\Theta_\chi\|^2 = R_{\chi^-}\|\Theta_\chi\|^2 + 4(M^4/16)r_{\text{in}}^{-4}\|\nabla_\chi\Theta_\chi\|^2 + 4\|\Theta_\chi\|^2\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln r_{\text{in}}\right\rangle_{\chi^-} + 4\left\langle\nabla\ln r_{\text{in}}, \nabla\|\Theta_\chi\|^2\right\rangle_{\chi^-} \tag{104}$$

Finally Eq. (79) gives

$$\begin{aligned}2\Delta_{\chi^-}\|\Theta_\chi\|^2 &= 4(c^4/16)r_{\text{in}}^{-4}\|\nabla_\chi\Theta_\chi\|^2 + 4\|\Theta_\chi\|^2\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln r_{\text{in}}\right\rangle_{\chi^-} + 4\left\langle\nabla\ln r_{\text{in}}, \nabla\|\Theta_\chi\|^2\right\rangle_{\chi^-}\\ &+ \sigma^{-2}(\zeta^-)^{-1}\left(\mathcal{P} - Q_{\overline{g}} - U^{-1}\overline{\Delta}U + \frac{1}{2}|\overline{\nabla}\ln U|^2 - \left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln f^-\right\rangle\right)\|\Theta_\chi\|^2\end{aligned} \tag{105}$$

Identity for Σ^- is given in the following lemma.

Lemma 9.3.

$$\overline{\nabla}_{\overline{\chi^-}}\left(\sin^2\theta\left(2\overline{\nabla}\|\Theta_\chi\|^2 - 2\|\Theta_\chi\|^2\overline{\nabla}\ln r_{in} + 2|U|^{-3/4}\overline{\nabla}\ln r_{in} + \|\Theta_\chi\|^2\overline{\nabla}\ln U\right) - (\|\Theta_\chi\|^2-1)\overline{\nabla}\sin^2\theta\right)$$
$$= \left(4(c^4/16)r_{in}^{-4}\sigma^2\zeta^-(\mathcal{R}+iI) - (3/2)L^-\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln f^-\right\rangle + i(3/4)|U|^{-3/4}\left\langle\overline{\nabla}\omega, \overline{\nabla}\ln f^-\right\rangle\right)\sigma^{-2}(\zeta^-)^{-1}\sin^2\theta$$
$$+ \sigma^{-2}(\zeta^-)^{-1}\left(\mathcal{P} - Q_{\overline{g}} - (1/2)\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln U\right\rangle - (1/4)(4-3L^-)\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln f^-\right\rangle\right)\|\Theta_\chi\|^2\sin^2\theta$$
$$- 2\|\Theta_\chi\|^2\sigma^{-2}(\zeta^-)^{-1}\overline{g}^{\theta\theta}\cos(2\theta) + \overline{\Delta}_{\chi^-}\sin^2\theta \qquad (106)$$

where I is as in Lemma 9.2 and $\mathcal{R} = |U|^{-1}\|\nabla_\gamma\Theta_\gamma\|^2 \geq 0$.

Proof. Writing the 3-Laplacian Δ_{χ^-} relative to $\chi^- = \sigma^2\zeta^-\overline{g} + Uf^-d\phi^2$ in terms of the Laplacian of the 2-metric $\overline{\chi^-} = \sigma^2\zeta^-\overline{g}$ we get $2\Delta_{\chi^-}\|\Theta_\chi\|^2 = 2\overline{\Delta}_{\overline{\chi^-}}\|\Theta_\chi\|^2 + \left\langle\overline{\nabla}\ln(Uf^-), \overline{\nabla}\|\Theta_\chi\|^2\right\rangle_{\overline{\chi^-}}$. We also have $U^{-1}\|\Theta_\chi\|^2\overline{\Delta}_{\overline{\chi^-}}U = \overline{\nabla}_{\overline{\chi^-}}\left(\|\Theta_\chi\|^2U^{-1}\overline{\nabla}U\right) - \left\langle\overline{\nabla}\|\Theta_\chi\|^2, \overline{\nabla}\ln U\right\rangle_{\overline{\chi^-}} + \left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln U\right\rangle_{\overline{\chi^-}}\|\Theta_\chi\|^2$. So Eq. (105) gives

$$2\overline{\Delta}_{\overline{\chi^-}}\|\Theta_\chi\|^2 + \left\langle\overline{\nabla}\ln f^-, \overline{\nabla}\|\Theta_\chi\|^2\right\rangle_{\overline{\chi^-}} + \overline{\nabla}_{\overline{\chi^-}}\left(\|\Theta_\chi\|^2U^{-1}\overline{\nabla}U\right) = 4(c^4/16)r_{\text{in}}^{-4}\|\nabla_\chi\Theta_\chi\|^2 + 4\|\Theta_\chi\|^2\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln r_{\text{in}}\right\rangle_{\overline{\chi^-}}$$
$$+ 4\left\langle\overline{\nabla}\ln r_{\text{in}}, \nabla\|\Theta_\chi\|^2\right\rangle_{\overline{\chi^-}} + \sigma^{-2}(\zeta^-)^{-1}\left(\mathcal{P} - Q_{\overline{g}} - (1/2)\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln U\right\rangle - \left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln f^-\right\rangle\right)\|\Theta_\chi\|^2 \Rightarrow$$

$$2\overline{\Delta}_{\overline{\chi^-}}\|\Theta_\chi\|^2 - 2\left\langle\overline{\nabla}\ln r_{\text{in}}, \overline{\nabla}\|\Theta_\chi\|^2\right\rangle_{\overline{\chi^-}} + \overline{\nabla}_{\overline{\chi^-}}\left(\|\Theta_\chi\|^2U^{-1}\overline{\nabla}U\right) = 4(c^4/16)r_{\text{in}}^{-4}\|\nabla_\chi\Theta_\chi\|^2 + 4\|\Theta_\chi\|^2\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln r_{\text{in}}\right\rangle_{\overline{\chi^-}}$$
$$- \left\langle\overline{\nabla}\ln\sin^2\theta, \overline{\nabla}\|\Theta_\chi\|^2\right\rangle_{\overline{\chi^-}} + \sigma^{-2}(\zeta^-)^{-1}\left(\mathcal{P} - Q_{\overline{g}} - (1/2)\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln U\right\rangle - \left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln f^-\right\rangle\right)\|\Theta_\chi\|^2 \Rightarrow$$

$$2\overline{\Delta}_{\overline{\chi^-}}\|\Theta_\chi\|^2 + \overline{\nabla}_{\overline{\chi^-}}\left(-2\|\Theta_\chi\|^2\overline{\nabla}\ln r_{\text{in}} + 2|U|^{-\frac{3}{4}}\overline{\nabla}\ln r_{\text{in}} + \|\Theta_\chi\|^2\overline{\nabla}\ln U\right) =$$
$$4(c^4/16)r_{\text{in}}^{-4}\|\nabla_\chi\Theta_\chi\|^2 - (3/2)|U|^{-3/4}\left\langle\overline{\nabla}\ln|U|, \overline{\nabla}\ln r_{\text{in}}\right\rangle_{\overline{\chi^-}} + 4\|\Theta_\chi\|^2\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln r_{\text{in}}\right\rangle_{\overline{\chi^-}}$$
$$- \left\langle\overline{\nabla}\ln\sin^2\theta, \overline{\nabla}\|\Theta_\chi\|^2\right\rangle_{\overline{\chi^-}} + \sigma^{-2}(\zeta^-)^{-1}\left(\mathcal{P} - Q_{\overline{g}} - (1/2)\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln U\right\rangle - \left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln f^-\right\rangle\right)\|\Theta_\chi\|^2$$

Multiplying both sides by $\sin^2\theta$ and using that U, r_{in} are functions of r only we get

$$\overline{\nabla}_{\overline{\chi^-}}\left(\sin^2\theta\left(2\overline{\nabla}\|\Theta_\chi\|^2 - 2\|\Theta_\chi\|^2\overline{\nabla}\ln r_{\text{in}} + 2|U|^{-3/4}\overline{\nabla}\ln r_{\text{in}} + \|\Theta_\chi\|^2\overline{\nabla}\ln U\right)\right) - 2\left\langle\overline{\nabla}\sin^2\theta, \overline{\nabla}\|\Theta_\chi\|^2\right\rangle_{\overline{\chi^-}}$$
$$= 4\sin^2\theta(c^4/16)r_{\text{in}}^{-4}\|\nabla_\chi\Theta_\chi\|^2 - (3/2)\sin^2\theta|U|^{-3/4}\left\langle\overline{\nabla}\ln|U|, \overline{\nabla}\ln r_{\text{in}}\right\rangle_{\overline{\chi^-}} + 4\sin^2\theta\|\Theta_\chi\|^2\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln r_{\text{in}}\right\rangle_{\overline{\chi^-}}$$
$$- \left\langle\overline{\nabla}\sin^2\theta, \overline{\nabla}\|\Theta_\chi\|^2\right\rangle_{\overline{\chi^-}} + \sigma^{-2}(\zeta^-)^{-1}\left(\mathcal{P} - Q_{\overline{g}} - (1/2)\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln U\right\rangle - \left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln f^-\right\rangle\right)\|\Theta_\chi\|^2\sin^2\theta \Rightarrow$$

$$\overline{\nabla}_{\overline{\chi^-}}\left(\sin^2\theta\left(2\overline{\nabla}\|\Theta_\chi\|^2 - 2\|\Theta_\chi\|^2\overline{\nabla}\ln r_{\text{in}} + 2|U|^{-3/4}\overline{\nabla}\ln r_{\text{in}} + \|\Theta_\chi\|^2\overline{\nabla}\ln U\right)\right) - \overline{\nabla}_{\overline{\chi^-}}\left(\|\Theta_\chi\|^2\nabla\sin^2\theta\right)$$
$$= 4\sin^2\theta(c^4/16)r_{\text{in}}^{-4}\|\nabla_\chi\Theta_\chi\|^2 - (3/2)\sin^2\theta|U|^{-3/4}\left\langle\overline{\nabla}\ln|U|, \overline{\nabla}\ln r_{\text{in}}\right\rangle_{\overline{\chi^-}} + 4\sin^2\theta\|\Theta_\chi\|^2\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln r_{\text{in}}\right\rangle_{\overline{\chi^-}} +$$
$$\sigma^{-2}(\zeta^-)^{-1}\left(\mathcal{P} - Q_{\overline{g}} - (1/2)\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln U\right\rangle - \left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln f^-\right\rangle\right)\|\Theta_\chi\|^2\sin^2\theta - \|\Theta_\chi\|^2\overline{\Delta}_{\chi^-}\sin^2\theta \Rightarrow$$

$$
\begin{aligned}
&\overline{\nabla}_{\overline{\chi^-}}\left(\sin^2\theta\left(2\overline{\nabla}\|\Theta_\chi\|^2 - 2\|\Theta_\chi\|^2\overline{\nabla}\ln r_{\mathrm{in}} + 2|U|^{-3/4}\overline{\nabla}\ln r_{\mathrm{in}} + \|\Theta_\chi\|^2\overline{\nabla}\ln U\right) - (\|\Theta_\chi\|^2 - 1)\overline{\nabla}\sin^2\theta\right)\\
&\quad = \sigma^{-2}(\zeta^-)^{-1}\left(4(c^4/16)r_{\mathrm{in}}^{-4}\sigma^2\zeta^-\|\nabla_\chi\Theta_\chi\|^2 - (3/4)|U|^{-3/4}\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln f^-\right\rangle + i(3/4)|U|^{-3/4}\left\langle\overline{\nabla}\omega, \overline{\nabla}\ln f^-\right\rangle + \right.\\
&4\|\Theta_\chi\|^2\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln r_{\mathrm{in}}\right\rangle + \left.\left(\mathcal{P} - Q_{\overline{g}} - (1/2)\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln U\right\rangle - \left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln f^-\right\rangle\right)\|\Theta_\chi\|^2\right)\sin^2\theta - (\|\Theta_\chi\|^2 - 1)\overline{\Delta}_{\chi^-}\sin^2\theta
\end{aligned}
$$

Using $|U|^{-3/4} = L^-\|\Theta_\chi\|^2$ and Eq. (96) we get

$$
\begin{aligned}
&\overline{\nabla}_{\overline{\chi^-}}\left(\sin^2\theta\left(2\overline{\nabla}\|\Theta_\chi\|^2 - 2\|\Theta_\chi\|^2\overline{\nabla}\ln r_{\mathrm{in}} + 2|U|^{-3/4}\overline{\nabla}\ln r_{\mathrm{in}} + \|\Theta_\chi\|^2\overline{\nabla}\ln U\right) - (\|\Theta_\chi\|^2 - 1)\overline{\nabla}\sin^2\theta\right)\\
&\quad = \sigma^{-2}(\zeta^-)^{-1}\left(4(c^4/16)r_{\mathrm{in}}^{-4}\sigma^2\zeta^-\|\nabla_\chi\Theta_\chi\|^2 - (3/2)L^-\|\Theta_\chi\|^2\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln f^-\right\rangle + i(3/4)|U|^{-3/4}\left\langle\overline{\nabla}\omega, \overline{\nabla}\ln f^-\right\rangle + \right.\\
&\quad 4\|\Theta_\chi\|^2\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln r_{\mathrm{in}}\right\rangle + \left.\left(\mathcal{P} - Q_{\overline{g}} - (1/2)\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln U\right\rangle - (1/4)(4 - 3L^-)\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln f^-\right\rangle\right)\|\Theta_\chi\|^2\right)\sin^2\theta\\
&\qquad - 2\|\Theta_\chi\|^2\sigma^{-2}(\zeta^-)^{-1}\overline{g}^{\theta\theta}\cos(2\theta) + \overline{\Delta}_{\chi^-}\sin^2\theta
\end{aligned}
$$

Since $4\sigma^{-2}(\zeta^-)^{-1}\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln r_{\mathrm{in}}\right\rangle = -2\sigma^{-2}(\zeta^-)^{-1}\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln f\right\rangle = -2(c^4/16)r_{\mathrm{in}}^{-4}\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln r_{\mathrm{in}}\right\rangle_{\overline{\chi}}$ we get Eq. (106) using Lemma 9.2. □

10 Main theorem

We separate the positive part of the contribution of the integral of $\|\Theta_\chi\|^2\sigma^{-2}(\zeta^\pm)^{-1}\overline{g}^{\theta\theta}\cos(2\theta)$ on $\Sigma^\pm$. Although this integral will be zero in case $\|\Theta_\chi\|^2$ is constant the integrand is not identically zero even in the case of Kerr-Newman solution. This follows from Eq. (96). Let

$$
\mathcal{A}(r_0) = \frac{\int\limits_{r=r_0}\sigma^{3/2}\|\Theta_\vartheta\|^2\cos(2\theta)d\theta}{\int\limits_{r=r_0}\sigma^{3/2}\|\Theta_\vartheta\|^2\sin^2\theta d\theta} \tag{107}
$$

As usual we are interested only for $r_0 > M + c$. We recall that $\sigma^{3/2}\|\Theta_\vartheta\|^2 = X_{\mathrm{K}}^{1/8}\zeta^{-1/4}X^{5/8}f^{-3/4}\|\Theta_{\overline{g}}\|^2 = X_{\mathrm{K}}^{1/8}O\left((\sin\theta)^{-1/4}\right)$ which is integrable. We show

$$
\int\limits_{(r_1,r_2),\overline{\chi^\pm}}\|\Theta_\chi\|^2\sigma^{-2}(\zeta^\pm)^{-1}\overline{g}^{\theta\theta}\cos(2\theta) = \int\limits_{(r_1,r_2),\overline{\chi^\pm}}\sin^2\theta\|\Theta_\chi\|^2\sigma^{-2}(\zeta^\pm)^{-1}\overline{g}^{\theta\theta}\mathcal{A} \tag{108}
$$

LHS$= \iint\limits_{(r_1,r_2)}|U|^{-3/4}\|\Theta_\vartheta\|^2\sigma^{3/2-2}(\zeta^\pm)^{-1}\overline{g}^{\theta\theta}\cos(2\theta)\sqrt{\chi^\pm_{rr}\chi^\pm_{\theta\theta}}drd\theta$
$= \iint\limits_{(r_1,r_2)}|U|^{-3/4}\sigma^{3/2}\|\Theta_\vartheta\|^2\cos(2\theta)(r^2 - 2M'r + a^2)^{-1/2}drd\theta$
$= \int_{r_1}^{r_2}|U|^{-3/4}(r^2 - 2M'r + a^2)^{-1/2}\left(\int\sigma^{3/2}\|\Theta_\vartheta\|^2\cos(2\theta)d\theta\right)dr$
$= \int_{r_1}^{r_2}|U|^{-3/4}(r^2 - 2M'r + a^2)^{-1/2}\mathcal{A}\left(\int\sigma^{3/2}\|\Theta_\vartheta\|^2\sin^2\theta d\theta\right)dr$

$= \iint\limits_{(r_1,r_2)} \sin^2\theta |U|^{-3/4}\sigma^{3/2}\|\Theta_\vartheta\|^2(r^2-2M'r+a^2)^{-1/2}\mathcal{A} dr d\theta$
$= \iint\limits_{(r_1,r_2)} \sin^2\theta |U|^{-3/4}\sigma^{3/2}\|\Theta_\vartheta\|^2(r^2-2M'r+a^2)^{-1/2}\mathcal{A}\sqrt{\chi^{rr}\chi^{\theta\theta}}\sqrt{\chi_{rr}\chi_{\theta\theta}} dr d\theta$
$= \int\limits_{(r_1,r_2),\overline{\chi^\pm}} \sin^2\theta \|\Theta_\chi\|^2\sigma^{-2}(\zeta^\pm)^{-1}\overline{g}^{\theta\theta}\mathcal{A}$. Having checked Eq. (108) we now decompose $\mathcal{A}$ into positive and negative parts. Positive part we add with $Q_{\overline{g}}$ since there is a negative sign in front of $2\|\Theta_\chi\|^2(\zeta^-)^{-1}\overline{g}^{\theta\theta}\cos(2\theta)$ in Eqs. (94,106). We define

$$\widehat{Q_{\overline{g}}} = Q_{\overline{g}} + 2\overline{g}^{\theta\theta}\mathcal{A}^+ \tag{109}$$

$\widehat{Q_{\overline{g}}}$ is nonnegative. This will be important in the proof of the main theorem. We define $Q_\Pi = \left(4\left\langle \overline{\nabla}\ln\left(X_{\mathrm{K}}/\sqrt{\rho}\right), \overline{\nabla}\ln\sigma\right\rangle_\Pi + \overline{g}_{\theta\theta}f_{\mathrm{em}}\right)^-$. As explained before Lemma 7.1, Q_Π is well-defined for $r > M + c$. Then from Eqs. (61,62,69) we get $Q_{\overline{g}} = \overline{g}^{\theta\theta}Q_\Pi$. Similarly we write Eq. (109) as

$$\widehat{Q_{\overline{g}}} = \overline{g}^{\theta\theta}\widehat{Q_\Pi} \tag{110}$$

where $\widehat{Q_\Pi} = Q_\Pi + 2\mathcal{A}^+$. We now define an average of $\widehat{Q_\Pi}$ on a r = constant loop on a ϕ = constant surface.

$$Q_{\mathrm{ave}}(r_0) = \frac{\int\limits_{r=r_0}\sigma^{3/2}\widehat{Q_\Pi}\|\Theta_\vartheta\|^2\sin^2\theta d\theta}{\int\limits_{r=r_0}\sigma^{3/2}\|\Theta_\vartheta\|^2\sin^2\theta d\theta} \tag{111}$$

Thus $Q_{\mathrm{ave}}(r_0) = \frac{\int\limits_{r=r_0}\sigma^{3/2}Q_\Pi\|\Theta_\vartheta\|^2\sin^2\theta d\theta}{\int\limits_{r=r_0}\sigma^{3/2}\|\Theta_\vartheta\|^2\sin^2\theta d\theta} + 2\mathcal{A}^+$. Since $\int\limits_{r=r_0}\sigma^{3/2}\|\Theta_\vartheta\|^2\cos(2\theta)d\theta =$ $\int\limits_{r=r_0}\left(1+O(r^{-1})\right)\cos(2\theta)d\theta = O(r^{-1})$ as $r\to\infty$, we have

$$Q_{\mathrm{ave}}(r) = O(r^{-1}) \tag{112}$$

Also as $\sin\theta \to 0$ and $r \downarrow M + c$, $Q_{\mathrm{ave}}(r)$ is bounded because Q_Π is at worst $O\left((\sin\theta)^{-2}\right)$ and $\sigma^{3/2}\|\Theta_\vartheta\|^2 = X_{\mathrm{K}}^{1/8}O\left((\sin\theta)^{-1/4}\right)$. In fact we have

$$4\left\langle\overline{\nabla}\ln(X_{\mathrm{K}}/\sqrt{\rho}), \overline{\nabla}\ln\sigma\right\rangle = \begin{cases} \overline{g}^{\theta\theta}O(r^{-2}) \text{ as } r\to\infty, \\ \overline{g}^{\theta\theta}O(\sin\theta) \text{ as } \theta\to 0 \text{ or } \pi, \\ \overline{g}^{\theta\theta}O(1) \text{ as } r\to M+c,\ \theta\neq 0,\pi \end{cases}$$

where, we recall from Eq. (62), that $\overline{g}^{\theta\theta} = \Omega^{-1}(r^2 - 2M'r + (M^2 - \mathfrak{e}^2 - \mathfrak{m}^2)\sin^2\theta + a^2\cos^2\theta)^{-1}$. $\overline{g}^{\theta\theta}$ is well defined for $r > M + c$. This factor does not occur in the

inner product relative to the metric Π and hence in Q_Π and Q_{ave}. Also $Q_{\text{ave}}(r)$ is not defined at isolated points where $\int_{r=\hat{r}} \sigma^{3/2}\|\Theta_\vartheta\|^2 \sin^2\theta d\theta = 0$ but these are removable and $Q_{\text{ave}}(r)$ remains continuous after redefinition. Such a value exists only if the known spinor $\Theta_{\widehat{g}}$ vanishes on the entire $r = \hat{r}$ loop.

Lemma 10.1. *Suppose $U = U(r)$ and*

$$\left(r^2 - 2M'r + a^2\right)\left(\frac{d\ln U}{dr}\right)^2 + (4 - 3L_{ave})\sqrt{r^2 - 2M'r + a^2}\frac{d\ln U}{dr} + 2Q_{ave} = 0 \tag{113}$$

$$\textit{Then} \iint_{\Sigma_2^+,\overline{\chi}} \left(2\langle\nabla\ln U, \nabla\ln U\rangle + (4-3L)\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln\ f\right\rangle + 4\widehat{Q_{\overline{g}}}\right)\sigma^{-2}\zeta^{-1}\sin^2\theta\|\Theta_\chi\|^2 = 0 \tag{114}$$

Proof. Using Eqs. (89,57,52) we see that the LHS of Eq. (114) is
$\iint_{(r_1,r_2)} \sin^2\theta\left(2\left(r^2 - 2M'r + a^2\right)\left(\frac{d\ln U}{dr}\right)^2 + (4-3L)\left(r^2 - 2M'r + a^2\right)\frac{d\ln U}{dr}\frac{\partial\ln f}{\partial r} + 4\widehat{Q_\Pi}\right)\overline{g}^{\theta\theta}\sigma^{-2}\zeta^{-1}\|\Theta_\chi\|^2\sqrt{\chi_{rr}\chi_{\theta\theta}}drd\theta$
$= \iint_{(r_1,r_2)} \sin^2\theta\left(2\left(r^2 - 2M'r + a^2\right)\left(\frac{d\ln U}{dr}\right)^2 + (4-3L)\left(r^2 - 2M'r + a^2\right)\frac{d\ln U}{dr}\frac{\partial\ln f}{\partial r} + 4\widehat{Q_\Pi}\right)|U|^{-3/4}\sigma^{3/2}\|\Theta_\vartheta\|^2$
$(r^2 - 2M'r + a^2)^{-1/2}drd\theta.$ We use the definition of Q_{ave} given in Eq. (111). Also from (93) and noting that $(\partial\ln f/\partial r) = 2(r^2 - 2M'r + a^2)^{-1/2}$ we get

$$\int L\sigma^{3/2}\left(r^2 - 2M'r + a^2\right)\frac{d\ln U}{dr}\frac{\partial\ln f}{\partial r}\|\Theta_\vartheta\|^2\sin^2\theta d\theta = 2L_{\text{ave}}\sqrt{r^2 - 2M'r + a^2}\frac{d\ln U}{dr}\int\sigma^{3/2}\|\Theta_\vartheta\|^2\sin^2\theta d\theta$$

So replacing $\widehat{Q_\Pi}$ and L by their respective average values Q_{ave} and L_{ave} we write the LHS of Eq. (114) as
$\iint_{(r_1,r_2)} \sin^2\theta\left(2\left(r^2 - 2M'r + a^2\right)\left(\frac{d\ln U}{dr}\right)^2 + 2(4 - 3L_{\text{ave}})\sqrt{r^2 - 2M'r + a^2}\frac{d\ln U}{dr} + 4Q_{\text{ave}}\right)|U|^{-3/4}\sigma^{3/2}\|\Theta\|^2$
$(r^2 - 2M'r + a^2)^{-1/2}drd\theta$ which is 0 by Eq. (113). □

Thus if U is a solution of (113) then integrating (94) we get

$$\iint_{\Sigma_2^+,\overline{\chi}}\left(\sin^2\theta\left(4\|\nabla_\chi\Theta_\chi\|^2 + \|\Theta_\chi\|^2\sigma^{-2}\zeta^{-1}\left(\mathcal{P} + 2\mathcal{A}^-\overline{g}^{\theta\theta} - (3/4)iL\left\langle\overline{\nabla}\omega, \overline{\nabla}\ln f\right\rangle\right)\right)\right) = \lim_{r_1\to M+c}\int_{r=r_1}\mathscr{B}d\theta + \lim_{r_2\to\infty}\int_{r=r_2}\mathscr{B}d\theta$$

where we wrote the boundary integrands with respect to outward normals by $\mathscr{B}$. Before computing $\mathscr{B}$ we cannot give rigorous details. However we state roughly our plan. The part of the axis given by $\theta = 0$ or $\theta = \pi$ alone (alone means in the region $r > M + c$ as explained after Eq. (46)) does not appear as a limiting boundary. If we consider a θ = constant = θ_0 line for θ_0 close to 0 or π as a boundary then the LHS of Eq. (94) gives on this boundary $\int_{\theta_0}\left\langle 2\sin^2\theta\overline{\nabla}\|\Theta_\chi\|^2 - (\|\Theta_\chi\|^2 - 1)\overline{\nabla}\sin^2\theta, \overline{n}\right\rangle_{\overline{\chi}}$ where $\overline{n}$ is the unit normal form on this line. All other terms are orthogonal to $\overline{n}$. The integral vanishes as $\theta_0 \to 0$ or π because it is bounded by $C\sin\theta\int(r^2 - 2M'r + a^2)^{-1/2}r^{-1}dr$

for some constant C. r^{-1} factor comes from $\|\Theta_\chi\|^2 - 1$ for large values of r. This factor is necessary only to make the constant C independent of r. But we can also take the $\theta_0 \to 0$ or π limit for a fixed large r first and then let $r \to \infty$. The boundary term from $\overline{\Delta}_{\overline{\chi}} \sin^2\theta$ in the RHS of Eq. (94) vanishes.

Now we want to compute the boundary integrals. But in order to evaluate the boundary integral at $r = M + c$ and not on the axis we are forced to consider the identity in Eq. (106) on Σ^- and to match U suitably across the smooth parts of the limiting surface at $r = M + c$. We proceed as follows.

A solution of Eq. (113) for $4 - 3L_{\text{ave}} \geq 0$ is given by the first case of the following equation.

$$\frac{d\ln U}{dr} = \begin{cases} \dfrac{-2 + (3/2)L_{\text{ave}} + \sqrt{(2-(3/2)L_{\text{ave}})^2 - 2Q_{\text{ave}}}}{\sqrt{r^2 - 2M'r + a^2}} & \text{if } 4 - 3L_{\text{ave}} \geq 0, \\ i\sqrt{2Q_{\text{ave}}/(r^2 - 2M'r + a^2)} & \text{otherwise} \end{cases} \tag{115}$$

For this solution for $r > M + c$, $\operatorname{Re} \dfrac{d\ln U}{dr} \leq 0$. Writing for this solution

$$-\iint_{\Sigma_2^+, \overline{\chi}} \sigma^{-2}\zeta^{-1}\sin^2\theta\Big((1/2)\langle\nabla\ln U, \nabla\ln U\rangle + (1/4)(4-3L)\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln\ f\right\rangle + \widehat{Q_{\overline{g}}}\Big)\|\Theta_\chi\|^2 = i\iint_{\Sigma_2^+, \overline{\chi}} I_1\sigma^{-2}\zeta^{-1}\overline{g}^{\theta\theta}\sin^2\theta\|\Theta_\chi\|^2 \tag{116}$$

we see that I_1 is a real function on Σ^+. Absorbing the other pure imaginary (or zero) term of Eq. (94) we define

$$i\mathcal{I}_{\text{out}}\zeta^{-1} = iI_1\zeta^{-1}\overline{g}^{\theta\theta} - (3/4)\zeta^{-1}iL|U|^{-3/4}\left\langle\overline{\nabla}\omega, \overline{\nabla}\ln f\right\rangle \tag{117}$$

Similarly on Σ^- we seek U such that

$$2\langle\nabla\ln U, \nabla\ln U\rangle + (4 - 3L^-)\left\langle\overline{\nabla}\ln U, \overline{\nabla}\ln\ f^-\right\rangle + 4\widehat{Q_{\overline{g}}} = 0 \tag{118}$$

for $4 - 3L_{\text{ave}} \geq 0$.

Since $(\partial\ln f^-/\partial r) = -2(r^2 - 2M'r + a^2)^{-1/2}$ this time we need

$$\left(r^2 - 2M'r + a^2\right)\left(\frac{d\ln U}{dr}\right)^2 - (4 - 3L^-_{\text{ave}})\frac{d\ln U}{dr} + 2Q_{\text{ave}} = 0 \tag{119}$$

A solution of Eq. (119) for $4 - 3L^-_{\text{ave}} \geq 0$ is given by the first case of

$$\frac{d\ln U}{dr} = \begin{cases} \dfrac{2 - (3/2)L^-_{\text{ave}} - \sqrt{(2-(3/2)L^-_{\text{ave}})^2 - 2Q_{\text{ave}}}}{\sqrt{r^2 - 2Mr}} & \text{if } 4 - 3L^-_{\text{ave}} \geq 0, \\ -i\sqrt{2Q_{\text{ave}}/(r^2-2Mr)} & \text{otherwise} \end{cases} \tag{120}$$

For this solution for $r > M+c$, $\mathrm{Re}\,\dfrac{d\ln U}{dr} \geq 0$. In particular $\mathrm{Re}\, L^-\left\langle \overline{\nabla}\ln U, \overline{\nabla}\ln f^-\right\rangle \leq 0$. Thus in Eq. (106) for Σ^- we write

$$\iint\limits_{\Sigma_2^-,\overline{\chi^-}} \sigma^{-2}(\zeta^-)^{-1}\sin^2\theta\Big(4(c^4/16)r_{\text{in}}^{-4}\sigma^2\zeta^-(\mathcal{R}+iI) - (1/2)\langle\nabla\ln U,\nabla\ln U\rangle - (1/4)(4-3L^-)\left\langle\overline{\nabla}\ln U,\overline{\nabla}\ln f^-\right\rangle - \widehat{Q_{\overline{g}}}$$
$$-(3/2)L^-\left\langle\overline{\nabla}\ln U,\overline{\nabla}\ln f^-\right\rangle + i(3/4)L^-\left\langle\overline{\nabla}\omega,\overline{\nabla}\ln f^-\right\rangle\Big)||\Theta_\chi||^2 = \iint\limits_{\Sigma_2^-,\overline{\chi^-}}\sigma^{-2}(\zeta^-)^{-1}\sin^2\theta\,(\mathcal{R}_{\text{in}} + i\mathcal{I}_{\text{in}})\,||\Theta_\chi||^2 \tag{121}$$

where both $\mathcal{R}_{\text{in}}$ and $\mathcal{I}_{\text{in}}$ are real and $\mathcal{R}_{\text{in}} \geq 0$.

Since on Σ^-, $|U|^{-3/4} = L^-||\Theta_\chi||^2$ and $||\Theta_\chi||^2 = \sigma^{3/2}|U|^{-3/4}||\Theta_\vartheta||^2$ we have $\sigma^{3/2}L^-||\Theta_\vartheta||^2 = 1$. Since $||\Theta_\vartheta||^2$ is continuous across $r = M+c$, at $r = M+c$, $L^- = L$ and hence $L^-_{\text{ave}} = L_{\text{ave}}$. It now follows from Eqs. (115,120) that

$$\left.\frac{d\ln U}{dr_{\text{out}}}\right|_{r=M+c} = \left.\frac{d\ln U}{dr_{\text{in}}}\right|_{r=M+c} \tag{122}$$

The above equation holds even before we match U on both $\Sigma_2^\pm$ at $r = M+c$ using the constant of integration.

Lemma 10.2. *Let $n_{\overline{\chi}}$ be the unit normal form relative to the metric $\overline{\chi}$ on the loop $r = r_0$ and the corresponding vector points in the direction of decreasing r. Let $U = U(r)$. For $M + c < r_0 < \infty$,*

$$\oint\limits_{r_0,\overline{\chi}}\left\langle 2\sin^2\theta\overline{\nabla}||\Theta_\chi||^2 + 2\sin^2\theta||\Theta_\chi||^2\overline{\nabla}\ln r_{out} - 2\sin^2\theta|U|^{-3/4}\overline{\nabla}\ln r_{out} + \sin^2\theta||\Theta_\chi||^2\overline{\nabla}\ln U, n_{\overline{\chi}}\right\rangle_{\overline{\chi}} = \frac{1}{|U(r_0)|^{3/4}}$$
$$\int\limits_{r=r_0}\left(\sigma^{3/2}\left(||\Theta||^2\left((1/2)r_{out}\frac{d\ln U}{dr_{out}} - i(3/2)\sqrt{r^2-2M'r+a^2}\frac{d\omega}{dr} - 2\right) - 2\sqrt{r^2-2M'r+a^2}\frac{\partial||\Theta||^2}{\partial r}\right) + 2\right)\sin^2\theta d\theta \tag{123}$$

Similarly on Σ^- with $n_{\overline{\chi^-}}$ being the unit normal form relative to the metric $\overline{\chi^-}$ on the loop $r = r_0$ and the corresponding vector pointing in the direction of

decreasing r,

$$
\oint\limits_{r_0,\overline{\chi}} \left\langle 2\sin^2\theta\overline{\nabla}\|\Theta_\chi\|^2 + 2\sin^2\theta\|\Theta_\chi\|^2\overline{\nabla}\ln r_{in} - 2\sin^2\theta|U|^{-3/4}\overline{\nabla}\ln r_{in} + \sin^2\theta\|\Theta_\chi\|^2\overline{\nabla}\ln U, n_{\overline{\chi^-}}\right\rangle_{\chi^-} = -\frac{1}{|U(r_0)|^{3/4}}
$$
$$
\int\limits_{r=r_0}\left(\sigma^{3/2}\left(\|\Theta\|^2\left((1/2)r_{in}\frac{d\ln U}{dr_{in}} + i(3/2)\sqrt{r^2-2M'r+a^2}\frac{d\omega}{dr} - 2\right) + 2\sqrt{r^2-2M'r+a^2}\frac{\partial\|\Theta\|^2}{\partial r}\right) + 2\right)\sin^2\theta d\theta \tag{124}
$$

In case U is real near infinity, U^{-1} *is bounded and* $(d\ln U/dr) = o\left(r^{-1}\right)$ *as* $r \to \infty$, *right hand sides of Eqs. (123,124) vanish as* $r_0 \to \infty$.

Proof. First we note that $\overline{\nabla}\|\Theta_\chi\|^2 = \overline{\nabla}\left(\sigma^{3/2}|U|^{-3/4}\|\Theta_\vartheta\|^2\right) = \overline{\nabla}\left(\sigma^{3/2}U^{-3/4}e^{i(3/4)\omega}\|\Theta_\vartheta\|^2\right)$
$= \sigma^{3/2}e^{i(3/4)\omega}\left((3/2)U^{-3/4}\|\Theta_\vartheta\|^2\overline{\nabla}\ln\sigma - (3/4)U^{-3/4}\|\Theta_\vartheta\|^2\overline{\nabla}\ln U + i(3/4)U^{-3/4}\|\Theta_\vartheta\|^2\overline{\nabla}\omega + U^{-3/4}\overline{\nabla}\|\Theta_\vartheta\|^2\right)$
$= (3/2)\sigma^{3/2}|U|^{-3/4}\|\Theta_\vartheta\|^2\overline{\nabla}\ln\sigma - (3/4)\sigma^{3/2}|U|^{-3/4}\|\Theta_\vartheta\|^2\overline{\nabla}\ln U + i(3/4)\sigma^{3/2}|U|^{-3/4}\|\Theta_\vartheta\|^2\overline{\nabla}\omega + \sigma^{3/2}|U|^{-3/4}\overline{\nabla}\|\Theta_\vartheta\|^2$.
Now $n_{\overline{\chi},r} = -\sqrt{\chi_{rr}} = -\sigma\sqrt{\zeta g_{rr}}$. So for any differentiable function F on the $r = r_0$ loop, $\left\langle\overline{\nabla}F, n_{\overline{\chi}}\right\rangle_{\overline{\chi}}\sqrt{\overline{\chi}_{\theta\theta}} = \overline{\chi}^{rr}\frac{\partial F}{\partial r}n_{\overline{\chi},r}\sqrt{\overline{\chi}_{\theta\theta}} = -\sqrt{\overline{g}^{rr}}\frac{\partial F}{\partial r}\sqrt{\overline{g}_{\theta\theta}} = -\sqrt{r^2-2M'r+a^2}\frac{\partial F}{\partial r}$. Thus using $\|\Theta_\chi\|^2 = \sigma^{3/2}|U|^{-3/4}\|\Theta_\vartheta\|^2$ we get

$$
\oint\limits_{r_0,\overline{\chi}} \left\langle 2\sin^2\theta\overline{\nabla}\|\Theta_\chi\|^2 + 2\sin^2\theta\|\Theta_\chi\|^2\overline{\nabla}\ln r_{\mathrm{out}} - 2\sin^2\theta|U|^{-3/4}\overline{\nabla}\ln r_{\mathrm{out}} + \sin^2\theta\|\Theta_\chi\|^2\overline{\nabla}\ln U, n_{\overline{\chi}}\right\rangle_{\overline{\chi}} =
$$
$$
-\frac{\sqrt{r_0^2-2M'r_0+a^2}}{|U(r_0)|^{3/4}}\int\limits_{r=r_0}\left(\sigma^{3/2}\left(\|\Theta_\vartheta\|^2\left(-(1/2)\frac{d\ln U}{dr} + i(3/2)\frac{d\omega}{dr} + 2\frac{d\ln r_{\mathrm{out}}}{dr}\right) + 2\frac{\partial\|\Theta_\vartheta\|^2}{\partial r}\right) - 2\frac{d\ln r_{\mathrm{out}}}{dr}\right)\sin^2\theta d\theta =
$$
$$
\frac{1}{|U(r_0)|^{3/4}}\int\limits_{r=r_0}\left(\sigma^{3/2}\left(\|\Theta_\vartheta\|^2\left((1/2)r_{\mathrm{out}}\frac{d\ln U}{dr_{\mathrm{out}}} - i(3/2)\sqrt{r^2-2M'r+a^2}\frac{d\omega}{dr} - 2\right) - 2\sqrt{r^2-2M'r+a^2}\frac{\partial\|\Theta_\vartheta\|^2}{\partial r}\right) + 2\right)\sin^2\theta d\theta
$$

Similarly on Σ^- with $n_{\overline{\chi^-}}$ being the unit normal form relative to the metric $\overline{\chi^-}$ on the loop $r = r_0$ and the corresponding vector pointing in the direction of decreasing r,

$$
\oint\limits_{r_0,\overline{\chi}} \left\langle 2\sin^2\theta\overline{\nabla}\|\Theta_\chi\|^2 + 2\sin^2\theta\|\Theta_\chi\|^2\overline{\nabla}\ln r_{\mathrm{in}} - 2\sin^2\theta|U|^{-3/4}\overline{\nabla}\ln r_{\mathrm{in}} + \sin^2\theta\|\Theta_\chi\|^2\overline{\nabla}\ln U, n_{\overline{\chi^-}}\right\rangle_{\chi^-} =
$$
$$
-\frac{\sqrt{r_0^2-2M'r_0+a^2}}{|U(r_0)|^{3/4}}\int\limits_{r=r_0}\left(\sigma^{3/2}\left(\|\Theta_\vartheta\|^2\left(-(1/2)\frac{d\ln U}{dr} + i(3/2)\frac{d\omega}{dr} + 2\frac{d\ln r_{\mathrm{in}}}{dr}\right) + 2\frac{\partial\|\Theta_\vartheta\|^2}{\partial r}\right) - 2\frac{d\ln r_{\mathrm{in}}}{dr}\right)\sin^2\theta d\theta =
$$
$$
-\frac{1}{|U(r_0)|^{3/4}}\int\limits_{r=r_0}\left(\sigma^{3/2}\left(\|\Theta_\vartheta\|^2\left((1/2)r_{\mathrm{in}}\frac{d\ln U}{dr_{\mathrm{in}}} + i(3/2)\sqrt{r^2-2M'r+a^2}\frac{d\omega}{dr} - 2\right) + 2\sqrt{r^2-2M'r+a^2}\frac{\partial\|\Theta\|^2}{\partial r}\right) + 2\right)\sin^2\theta d\theta
$$

As $r_0 \to \infty$, $\|\Theta_\vartheta\|^2 - 1 = O(r^{-1})$. The contribution of the remaining terms vanish as $r_0 \to \infty$ by the hypotheses on U and Eq. (82). □

We now integrate Eqs. (94,106) on $\Sigma^{\pm}$. For both cases the boundary integrals at $r = M + c$ (equivalently $2r_{\rm in} = 2r_{\rm out} = c$) are evaluated w.r.t. to the normal vector pointing in the direction of decreasing r.

Lemma 10.3. *Suppose $U = U(r)$ is globally C^1 and U^{-1} is globally bounded and $(d\ln U/dr) = o(r^{-1})$ as $r \to \infty$ and U is a solution of Eq. (115) in Σ^+ and Eq. (120) in Σ^- such that $(d\ln U/dr)$ is locally bounded in $(M + c, \infty)$. and fails to be differentiable at most at finite number of points. Then*

$$\iint\limits_{\Sigma_2^+,\overline{\chi}} \left(\mathcal{P}\sigma^{-2}\zeta^{-1}\|\Theta_\chi\|^2 + 4\|\nabla_\chi\Theta_\chi\|^2 + \|\Theta_\chi\|^2\sigma^{-2}\zeta^{-1}\left(2\mathcal{A}^-\overline{g}^{\theta\theta} + i\mathcal{I}_{out}\right)\right)\sin^2\theta =$$
$$\lim_{r_0\downarrow M+c}\frac{1}{|U(r_0)|^{3/4}}\int\limits_{r=r_0}\left(\sigma^{3/2}\left(\|\Theta_\vartheta\|^2\left((1/2)r_{out}\frac{d\ln U}{dr_{out}} - i(3/2)\sqrt{r^2-2M'r+a^2}\frac{d\omega}{dr} - 2\right)\right) + 2\right)\sin^2\theta d\theta \quad (125)$$

The corresponding integral on $(\Sigma_2^-, \overline{\chi^-})$ is

$$\iint\limits_{\Sigma_2^-,\overline{\chi^-}} \left(\mathcal{P}\sigma^{-2}\zeta^{-1}\|\Theta_\chi\|^2 + 4\|\nabla_\chi\Theta_\chi\|^2 + \|\Theta_\chi\|^2\sigma^{-2}\zeta^{-1}\left(2\mathcal{A}^-\overline{g}^{\theta\theta} + \mathcal{R}_{in} + i\mathcal{I}_{in}\right)\right)\sin^2\theta =$$
$$-\lim_{r_0\downarrow M+c}\frac{1}{|U(r_0)|^{3/4}}\int\limits_{r=r_0}\left(\sigma^{3/2}\left(\|\Theta_\vartheta\|^2\left((1/2)r_{in}\frac{d\ln U}{dr_{in}} + i(3/2)\sqrt{r^2-2M'r+a^2}\frac{d\omega}{dr} - 2\right)\right) + 2\right)\sin^2\theta d\theta \quad (126)$$

The limiting sets for the integrals in the RHS of both equations are restricted to the horizon only.

Proof. Left hand sides result from the right hand sides of Eqs. (94,106) for the solutions U of Eqs. (115,120) after we modify $Q_{\overline{g}}$ to $\widehat{Q_{\overline{g}}}$ in Eq. (109) and replace the latter and L by their averages and use Eqs. (116,121). Right hand sides come from the right hand sides of Eqs. (123,124). We note that in the right hand sides of Eqs. (125,126) on the horizon $\sqrt{r^2 - 2M'r + a^2}\dfrac{\partial\|\Theta_\vartheta\|^2}{\partial r}$ drops out because of Eq. (84). Axis parts do not contribute in the limit $r_0 \downarrow M+c$. One way to see this is from Eq. (86). We see $\sigma^{3/2}\|\Theta_\vartheta\|^2\sin^2\theta d\theta = X_{\rm K}^{1/8}\zeta^{-1/4}X^{5/8}f^{-3/4}\|\Theta_{\overline{g}}\|^2\sin^2\theta d\theta = O\left((\sin\theta)^{-1/4}\right)\sin^2\theta d\theta$ and so differentiating partially relative to the variable r we get $\sigma^{3/2}(\partial\|\Theta_\vartheta\|^2/\partial r)\sin^2\theta d\theta = O\left((\sqrt{r^2-2M'r+a^2})^{-1}(\sin\theta)^{-1/4}\right)\sin^2\theta d\theta$. The factor $(\sqrt{r^2-2M'r+a^2})^{-1}$ comes from coordinate transformation Eqs. (44) when we change the partial derivative relative to r to partial derivatives relative to regular coordinates ρ, z and it get cancelled by the reciprocal factor in front of $(\partial\|\Theta_\vartheta\|^2/\partial r)$ in the left hand sides of Eqs. (123,124). Since on these parts of the r = constant surfaces $\sin\theta$ approaches 0 in the limit, taking the limit after integration results 0. Continuity of U and the ODE ensures the continuity of dlnU/dr and

hence gives the cancellations of the boundary integrals at finite number of values of r where U fails to be twice differentiable. □

If there are parts on the axis above the topmost and below the bottommost poles where $r \downarrow M+c$ and $\sin\theta \to 0$, the proof of the main theorem below becomes much more difficult and we shall need to consider the expression for $\Omega_{\rm K}$ in details. We include this expression which can be checked using Eqs. (39,60) and the formula after Eq. (63).

$$\Omega_{\rm K} = \varrho^2 \left((r-M-c)(r-M+c) + c^2\sin^2\theta\right)^{-1} \tag{127}$$

We recall $(r-M-c)(r-M+c) = r^2 - 2M'r + a^2$. Thus as $r \downarrow M+c$ and $\sin\theta \to 0$,

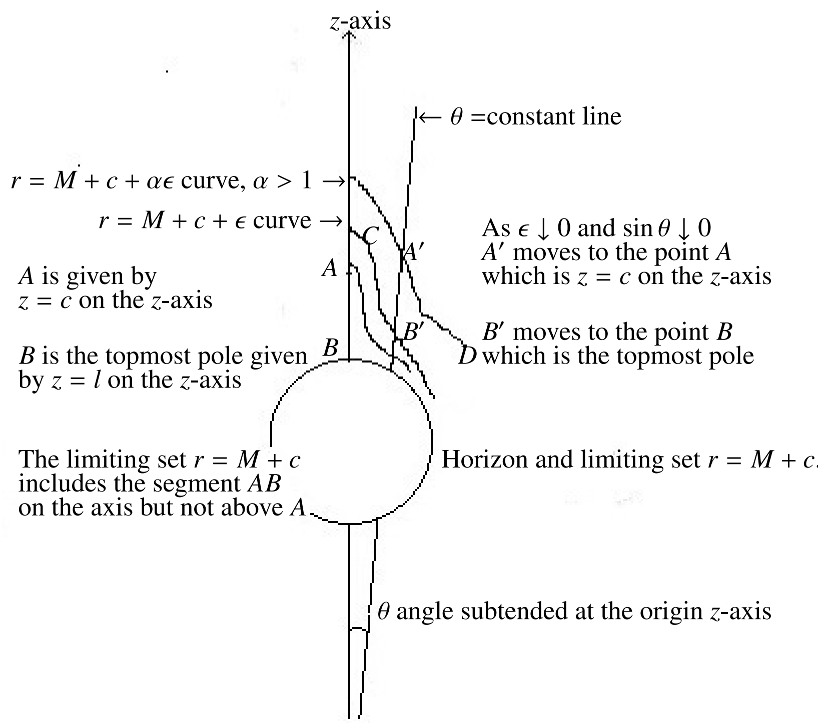
$$\begin{aligned}(\partial \ln \Omega_{\rm K}/\partial\theta) &= (\partial \ln \varrho^2/\partial\theta) - 2c^2(\sin\theta)(\cos\theta)(r^2 - 2M'r + a^2 + c^2\sin^2\theta)^{-1}\\ (\partial \ln \Omega_{\rm K}/\partial r) &= (\partial \ln \varrho^2/\partial\theta) - 2(r-M)(r^2 - 2M'r + a^2 + c^2\sin^2\theta)^{-1}\end{aligned}$$

We shall need to estimate the following two integrals respectively on a $\theta =$ constant curve and on the curve $r = M+c+\epsilon$ curve. Here $x = r - M, \alpha > 1$.

$$\begin{aligned}&\int_{M+c+\epsilon}^{M+c+\alpha\epsilon} (\partial \ln\Omega_{\rm K}/\partial\theta)(\sqrt{r^2-2M'r+a^2})^{-1} dr\\ &= -2c^2 \int_{c+\epsilon}^{c+2\epsilon} \frac{(\sin\theta)(\cos\theta)}{(x^2-c^2\cos^2\theta)\sqrt{x^2-c^2}} dx + O(\epsilon) = 2\left(\arctan\frac{x\tan\theta}{\sqrt{x^2-c^2}}\right)\Bigg|_{c+\epsilon}^{c+\alpha\epsilon} + O(\epsilon)\\ &= 2\arctan\frac{\sqrt{c/(2\alpha)}(1-\sqrt{\alpha})(\tan\theta)/\sqrt{\epsilon} + O(\sqrt{\epsilon})}{1 + (c/(2\sqrt{\alpha}) + O(\epsilon))((\tan\theta)/\sqrt{\epsilon})^2} + O(\sqrt{\epsilon}) = O((\sin\theta)/\sqrt{\epsilon})\end{aligned} \tag{128}$$

$$\begin{aligned}&\int_{\theta_1}^{\theta_2} (\partial \ln\Omega_{\rm K}/\partial r)\sqrt{r^2-2M'r+a^2} d\theta\\ &= -2x\int_{\theta_1}^{\theta_2} \frac{\sqrt{x^2-c^2}}{x^2-c^2+c^2\sin^2\theta} d\theta + O(\theta_2-\theta_1) = -2\arctan\frac{x\tan\theta}{\sqrt{x^2-c^2}}\Bigg|_{\theta_1}^{\theta_2} + O(\theta_2-\theta_1)\\ &= -2\arctan\frac{(c+\epsilon)\tan\theta}{\sqrt{2c\epsilon+\epsilon^2}}\Bigg|_{\theta_1}^{\theta_2} + O(\theta_2-\theta_1) = O((\sin\theta_1 - \sin\theta_2)/\sqrt{\epsilon})\end{aligned} \tag{129}$$

We need these integrals only near the limiting set as $r \downarrow M+c, \sin\theta \downarrow 0$ on the axis segment above the topmost and below the bottommost pole in case such segments have finite length. In the picture below we explain the axis segment above the topmost pole. It does not matter if the r = constant curves shown intersect the θ = constant more than once. The point is in case the segment AB



has finite length we can always find a $\theta = \theta(\alpha)$ such that $l_A \sin\theta$ = increment in $\rho = \int_c^{c+\alpha\epsilon} x(x^2 - c^2)^{-1/2} dx = \sqrt{2\alpha\epsilon} + O(\epsilon)$ and $l_B \sin\theta = \sqrt{2\epsilon} + O(\epsilon)$ where l_A, l_B are the z-cordinates of A, B.

Theorem 10.4. *There is no analytic multiple black hole solution of asymptotically flat stationary axisymmetric Einstein-Magnetic Maxwell equation with non-degenerate event horizon and the single black hole solution belongs to the Kerr-Newman family.*

Proof. First we recall that $L_{\text{ave}}, Q_{\text{ave}}$ are independent of U and are fixed functions. They are determined by the spacetime metric and the known spinor $\Theta_{\widehat{g}}$ which is also determined by the spacetime metric. $L_{\text{ave}} \to 1$ as $r \to \infty$. As $r \to \infty$, $Q_{\text{ave}} = O(r^{-1})$ by Eq. (112). Thus for large r, $(d \ln U/dr)$ is real and $\frac{d \ln U}{dr} = \frac{-1 + \sqrt{1 - O(r^{-1})}}{2\sqrt{r^2 - 2M'r + a^2}} = O(r^{-2})$. So near infinity U is real and U and its derivative have the appropriate decay needed for Lemma 10.3. Near $r = M + c$, U could be imaginary but its real and imaginary parts have the necessary decay. Near $r = M + c$, Q_{ave} is of the form $C_1 + C_2 r$ where C_1 and C_2 are real constants. Thus near $r = M + c$, U is an approximate solution of $\frac{d \ln U}{dr} = \frac{-C_3 - C_4 r}{\sqrt{r^2 - 2M'r + a^2}}$ for some complex constants C_3 and C_4. This gives $U \approx \exp\left(C_0 + \int \frac{-C_3 - C_4 r}{\sqrt{r^2 - 2M'r + a^2}} dr\right) = \exp\left(C_5 - C_4\sqrt{r^2 - 2M'r + a^2} - C_3 \ln\left(r - M + \sqrt{r^2 - 2M'r + a^2}\right)\right)$. U is never zero. Now the question is whether $(d \ln U/dr)$ can fail to be differentiable at infinite number of values of r. We already know the behavior of $(d \ln U/dr)$ near $r = M + c$ and for r sufficiently large. Away from the possible zero set of Θ in the region $r > M + c$ of 3-manifold, L is analytic. We consider the points in a compact r-interval $[M+c+\epsilon, b]$ where $\epsilon > 0$ at which radicals $\sqrt{(2 - (3/2)L_{\text{ave}})^2 - 2Q_{\text{ave}}}$ in Eq. (115) (and the corresponding one in Eq. (120)) vanish or $L_{\text{ave}} = 4/3$. These are the likely points where $(d \ln U/dr)$ can fail to be differentiable although it is regular in an open interval to the right. If $L_{\text{ave}}(\hat{r}) \leq 4/3$ then it follows its definition Eq. (93) that there is an interval about $\hat{r}$ such that $\int\limits_{r=\hat{r}} \sigma^{3/2} \|\Theta_\vartheta\|^2 \sin^2\theta d\theta > 0$ and this integral is analytic in this interval. So by Eq. (93), L_{ave} will be analytic in this interval. Thus if L_{ave} attains the value $4/3$ infinitely many times as $r \to \hat{r}$ then $L_{\text{ave}} = 4/3$ identically in an interval about $\hat{r}$. So we assume $L_{\text{ave}}(\hat{r}) < 4/3$ and ask whether the radicand can vanish infinitely many times as $r \to \hat{r}$. Similarly for $r > M + c, L_{\text{ave}}(\hat{r}) < 4/3$, in an analytic spacetime $\mathcal{A}(r)$ is an analytic function of r, and so $Q_{\text{ave}}(r)$ is an analytic function of r when $\mathcal{A}(r) > 0$. So assuming $L_{\text{ave}}(\hat{r}) < 4/3$, the radical $\sqrt{(2 - (3/2)L_{\text{ave}})^2 - 2Q_{\text{ave}}}$ can vanish only finite number of points. So $(d \ln U/dr)$ can fail to be differentiable only for finite number of values of r in the compact interval $[M + c + \epsilon, b]$. For $L_{\text{ave}}(\hat{r}) > 4/3$, $Q_{\text{ave}}(r)$ is not defined at isolated points where $\int\limits_{r=\hat{r}} \sigma^{3/2} \|\Theta_\vartheta\|^2 \sin^2\theta d\theta = 0$ but these are removable. Thus Lemma 10.3 applies and we can now add Eqs. (126,125) to find that the real parts of the LHS of both equations are zero by virtue of Eq. (122) provided we take the same value of $|U|$ at $M + c$ for U on both sides. The real part has nonnegative definite integrand. This shows $\mathcal{P} = 0$. $\mathcal{P} = 0$ gives $\sigma = 1$ identically. Thus by asymptotic conditions $X = X_{\text{K}}$. From $X = X_{\text{K}}$ alone we can-

not conclude that we have a single black hole. To claim that we have a single black hole we have to show that the 3-metrics $\widehat{g}$ and $\widehat{g}_{\mathrm{K}}$ are equal. We now prove this fact by showing $\Omega = \Omega_K$. Applying the conformal transformation formula to the metric $\overline{g} = \Omega\left(d\rho^2 + dz^2\right)$ we get $\overline{R} = -\overline{\Delta}\ln\Omega$. Similarly and using conformal invariance in 2-dimension we have $\overline{R}_{\mathrm{K}} = -\overline{\Delta}_{\overline{g}}\ln\Omega_{\mathrm{K}} = -(\Omega/\Omega_{\mathrm{K}})\overline{\Delta}\ln\Omega_{\mathrm{K}}$. But Eq. (37) gives $\overline{R} - (\Omega_{\mathrm{K}}/\Omega)\overline{R}_{\mathrm{K}} = -f_{\mathrm{em}}$. So we have $\overline{\Delta}\ln(\Omega_{\mathrm{K}}/\Omega) = -f_{\mathrm{em}}$ which is nonnegative because in addition to $\mathcal{P} = 0$ now we also have $f_{\mathrm{em}}^{+} = 0$. If we integrate this identity, the boundary integral at a $r = r_0 = $ constant $> M + c$ loop is (apart from the sign) $\int\limits_{r=r_0} (\partial\ln(\Omega_{\mathrm{K}}/\Omega)/\partial r)\sqrt{r^2 - 2M'r + a^2}d\theta$. By asymptotic condition $\ln(\Omega_{\mathrm{K}}/\Omega) = O(r^{-1})$ and $(\partial\ln(\Omega_{\mathrm{K}}/\Omega)/\partial r) = O(r^{-2})$ as $r \to \infty$. Thus the boundary integral goes to zero at ∞. Away from the axis the integral also goes to 0 as $r \downarrow M + c$ because near the horizon away from the poles we have $(\partial\ln\Omega/\partial r) = O(1)$ and away from the furthest poles and the adjacent axis segments where $r \downarrow M + c$ we have $(\partial\ln\Omega_{\mathrm{K}}/\partial r) = O(1)$. Near the poles $\ln\Omega = O(\ln\sin\theta)$ and hence $(\partial\ln\Omega/\partial r) = O(\ln\sin\theta)$ thus the contribution coming from $(\partial\ln\Omega/\partial r)$ near poles are $O\left(\int\limits_{\theta_1}^{\theta_2}\ln\sin\theta d\theta\right)$ which vanishes in the limit as both θ_1, θ_2 approaches the same value 0 or π. On the axis part given by $r \downarrow M + c$, $(\partial\ln\Omega_{\mathrm{K}}/\partial r)$ is not defined. In fact Ω_{K}^{-1} is zero on these line segments. So we shall integrate the identity $\overline{\Delta}\ln(\Omega_{\mathrm{K}}/\Omega) = -f_{\mathrm{em}}$ away from these line segments in the following way. In the limit $r \downarrow M + c$, the $r = $ constant curves are coming close to the inner parts of the axis (and possibly to parts of the axis adjacent to the topmost or bottommost poles) when there are more than one black hole. Before we approach each line segment between two neighboring black holes we take the boundary part to be a $\theta = $ constant line segment for θ close to 0 or π. On a $\theta = $ constant line segment the boundary integral will be (apart from the sign) $\int\limits_{\theta=\theta_0} (\partial\ln(\Omega_{\mathrm{K}}/\Omega)/\partial\theta)(\sqrt{r^2 - 2M'r + a^2})^{-1}dr$. Now we can arrange that the line $\theta = $ constant will intersect the $r = $ constant loop between two neighboring black holes even number of times. Thus this integral will be zero between to neighboring intersection points because limits of the integration will be the same. Between two neighboring intersection points where the $\theta = $ constant line is nearer to the axis we take the boundary to be the $r = $ constant loop. Boundary integral will vanish at this part because now we integrate relative to θ between two equal values of θ. The above argument does not apply for calculating the contribution of the boundary integral from the possible parts of the axis adjacent to the top-

most or bottommost poles where, in the limit $r \downarrow M + c$, a r = constant curve is coming close to the axis along an axis segment of nonzero length. Because now we cannot arrange that a θ = constant curve will intersect a r = constant curve in even number of times. That is without the absence of a black in one side this r = constant curve cannot cross the θ = constant curve to come out. In the following we only consider the segment over the topmost pole. In reference to the diagram we show why the length of the segment AB is zero. Suppose the length of AB is not zero. f_{em} is bounded for $r > M + c$. So on a set of negligible measure in this region the area integral of $\overline{\Delta}\ln(\Omega_{\mathrm{K}}/\Omega)$ is negligible. Thus the boundary line integral along $A'B'$ differs slightly from boundary integrals from the curves such as CB' and $A'D$ because on the bases CA' or $B'D$ of the almost triangular regions $CA'B'$ or $B'DA'$ the boundary contributions are becoming negligible as $\epsilon \to 0$. Thus the values of two integrals in Eqs. (128,129) can only differ slightly as $\epsilon \downarrow 0$ and $\sin\theta \downarrow 0$. But for the first integral Eq. (128) shows that the limit is not independent of α because $(\tan\theta)/\sqrt{\epsilon} = \sqrt{2\alpha}/l_A + O(\epsilon)$. So AB cannot have positive length and over the topmost pole we must have $r > M + c$. Thus the area integral of f_{em} is 0. Hence f_{em} is 0. Now Ω_{K}^{-1} is a differentiable function vanishing as $O((r - M - c)(r - M + c) + c^2 \sin^2\theta)$ at the poles and on the axis where $r \downarrow M + c$, whereas $\Omega = O((\sin^2\theta)^{-1}$ near the poles. We now integrate the identity $\overline{\Delta}(\Omega/\Omega_{\mathrm{K}}) = (\Omega_{\mathrm{K}}/\Omega)|\overline{\nabla}(\Omega/\Omega_{\mathrm{K}})|^2$. For the boundary integral with the domain of integration approaching the limiting set $r \downarrow M + c$ on the axis we follow the previous method of integration when both θ and r approach constant value on the same line segment. We now get $\Omega = \Omega_K$. So we have a single black hole and arguments of Bunting [3] or Mazur [4] give the uniqueness result. □

Remark 10.1. If f_{em} vanishes due to the absence of electromagnetic field we have the Kerr case. Now it is easier to show $W = W_{\mathrm{K}}$ (before proving $\Omega = \Omega_{\mathrm{K}}$) by exploiting one of Carter's equations namely $\overline{\nabla}\Big((X^2/\rho)\overline{\nabla}(W/X)\Big) = 0$. This equation gives following two identities since we have $X = X_{\mathrm{K}}$.

$$\overline{\nabla}\left(\frac{X^2}{\rho}\overline{\nabla}\frac{W - W_{\mathrm{K}}}{X}\right) = 0, \qquad \overline{\nabla}\left(\frac{(W - W_{\mathrm{K}})X^2}{X\rho}\overline{\nabla}\frac{W}{X}\right) = (X^2/\rho)\left|\overline{\nabla}\frac{W - W_{\mathrm{K}}}{X}\right|^2$$

Now on the axis $X^2(\partial(W/X)/\partial(\cos\theta)) = O(\rho^2)$. So the boundary integrals from the axis for $r > M+c$ in both identitities vanish. On the even horizon $X^2(\partial(W/X)/\partial(r)) = (\sin^2\theta)O(1)$. If the boundary integrals on the horizon as $r \downarrow M + c$ vanish for the first identity then they will also vanish for the second identity because $(W - W_{\mathrm{K}})/X$

is constant on each black hole surface. On the limiting set $r \downarrow M + c$ on the axis and between the black holes we follow the previous method of integration when both θ and r approach constant value on the same line segment. On the limiting set $r \downarrow M + c$ on the axis above the topmost or below the bottommost pole we integrate along a $r =$ constant curve. For example in the 1st identity the boundary contribution for these two line segments are proportional to $\int_{\theta_1}^{\theta_2} (X^2/\rho)(\partial((W - W_{\mathrm{K}})/X)/\partial r) \sqrt{r^2 - 2M'r + a^2} d\theta$ which goes to 0 as $\sin\theta_1, \sin\theta_2$ go to 0. Similarly in the second identity too ρ get cancelled and the integral tends to 0. Thus all the boundary integrals for the first and hence for both the identities vanish and we get $W = W_K$ from the second identity. So using $\rho^2 = VX + W^2$ we get $V = V_K$. Now it is well-known that if V, W are known then Ω can be solved uniquely using its asymptotic value. Finally we ask whether in the Kerr-Newman case one can prove $W = W_{\mathrm{K}}$ before proving $\Omega = \Omega_{\mathrm{K}}$. Assuming that W has the same positivity property as X one expects that by defining $\sigma = (W/W_{\mathrm{K}})^{1/4}$ one would be able to repeat the argument for showing $X = X_{\mathrm{K}}$.

11 Conclusion

We showed that spin-spin interaction cannot hold black holes apart in stationary equilibrium in an analytic asymptotically flat axisymmetric spacetime even in the presence of electromagnetic fields. The way we modified the method step by stem from the application of positive mass theorem in Bunting and Masood-ul-Alam [21] gives us hope that the method can be further modified to drop the axisymmetry assumption. This would however be a huge program because our method only shows $X = X_{\mathrm{K}}$. The equations to show the rest (including the equation for Ω) are changed without the assumption of axisymmetry. A more manageable problem the solution of which will also be a significant progress is as follows. In stead of Eq. (1) one starts with a non-axisymmetric spacetime metric of the form $-(V+\epsilon_{tt})dt^2 + 2(W+\epsilon_{t\phi})dtd\phi + 2\epsilon_{tx^A}dtdx^A + (X+\epsilon_{\phi\phi})d\phi^2 + \overline{g} + \epsilon_{\alpha\beta}dx^\alpha dx^\beta$ where ϵ is small and has appropriate boundary properties. In this case one expects that the error from the deviation from axisymmetry can be absorbed in a modified Q_{ave} having the required boundary properties so that our method would work.

12 Appendix

We outline the proofs of Lemma 8.1 and Lemma 9.2. If necessary further details can be found in Appendix I of [2]. We denote ∇_{g_1} by $\overset{1}{\nabla}$ and ∇_{g_2} by $\overset{2}{\nabla}$. Let $\{e(1)^i\}_{i=1,2,3}$ be orthonormal frame field of one forms for $g_1 = \overline{G} + f_1 d\phi^2$. Let $\{e(2)^i\}$ be the corresponding orthonormal one forms for $g_2 = \overline{G} + q f_1 d\phi^2$. We take $e(2)^\phi = \sqrt{q f_1} d\phi$ because $q f_1 \langle d\phi, d\phi \rangle_{g_2} = q f_1 g_2^{\phi\phi} = 1$. Similarly $e(1)^\phi = \sqrt{f_1} d\phi$. Thus $\sqrt{q} e(1)^\phi = e(2)^\phi$. Note $e(1)^A = e(2)^A$, $A = 1, 2$. Corresponding orthonormal frame field of vectors are $e(1)_A = e(2)_A$, $A = 1, 2$. $\dfrac{1}{\sqrt{q}} e(1)_\phi = e(2)_\phi$. $e(2)_A = \sqrt{\overline{G}^{11}} \dfrac{\partial}{\partial x^A}$, since $\langle e(2)_A, e(2)_B \rangle_{g_2} = \delta_{AB}$ but $\left\langle \dfrac{\partial}{\partial x^A}, \dfrac{\partial}{\partial x^A} \right\rangle_{g_2} = \overline{G}_{AB}$. However at a single point we can arrange $\overline{G}_{AB}$ to be δ_{AB}. We also note that $\left\langle e(2)_\phi, e(2)_\phi \right\rangle_{g_2} = q f_1 \left(e(2)_\phi\right)^\phi \left(e(2)_\phi\right)^\phi = f_1 \left(e(1)_\phi\right)^\phi \left(e(1)_\phi\right)^\phi = \left\langle e(1)^\phi, e(1)^\phi \right\rangle_{g_1}$. $\left\langle \dfrac{\partial}{\partial \phi}, \dfrac{\partial}{\partial \phi} \right\rangle_{g_2} = q f_1 \Rightarrow \overset{2}{e}_\phi = \dfrac{1}{\sqrt{q f_1}} \dfrac{\partial}{\partial \phi}$. $\left\langle \dfrac{\partial}{\partial \phi}, \dfrac{\partial}{\partial \phi} \right\rangle_{g_1} = f_1 \Rightarrow e(1)_\phi = \dfrac{1}{\sqrt{f_1}} \dfrac{\partial}{\partial \phi}$. First we compute the Christoffel symbols. Let Λ^C_{AD} be the Christoffel symbols of $\overline{G}$. Let $\overset{1}{\Gamma}{}^\mu_{\beta\gamma}$ and $\overset{2}{\Gamma}{}^\mu_{\beta\gamma}$ be the Christoffel symbols of g_1 and g_2. These symbols are w.r.t. $\left\{x^A, \phi\right\}$ coordinates. They are not the connection coefficients related to the one frame fields.

$$\overset{2}{\Gamma}{}^\phi_{\phi\phi} = 0,\ \overset{2}{\Gamma}{}^\phi_{AB} = 0,\ \overset{2}{\Gamma}{}^A_{B\phi} = 0,\ \overset{2}{\Gamma}{}^\phi_{\phi A} = \frac{1}{2}\frac{\partial \ln (q f_1)}{\partial x^A},\ \overset{2}{\Gamma}{}^A_{\phi\phi} = -\frac{1}{2}\frac{\partial (q f_1)}{\partial x^A},\ \overset{2}{\Gamma}{}^A_{BC} = \Lambda^A_{BC}$$

$$\overset{1}{\Gamma}{}^\phi_{\phi\phi} = 0,\ \overset{1}{\Gamma}{}^\phi_{AB} = 0,\ \overset{1}{\Gamma}{}^A_{B\phi} = 0,\ \overset{1}{\Gamma}{}^\phi_{\phi A} = \frac{1}{2}\frac{\partial \ln f_1}{\partial x^A},\ \overset{1}{\Gamma}{}^A_{\phi\phi} = -\frac{1}{2}\frac{\partial f_1}{\partial x^A},\ \overset{1}{\Gamma}{}^A_{BC} = \Lambda^A_{BC}$$

Now we calculate the connection coefficients $\overset{2}{C}$ and $\overset{1}{C}$ in the frame fields $\{e(2)_i\}$ and $\{e(1)_i\}$ for the two metrics respectively. On $\overset{2}{C}$ and $\overset{1}{C}$ the indices refer to the frame fields not coordinates.

$$\overset{1}{C}_{\phi\phi\phi} = \langle e(1)_\phi, \overset{1}{\nabla}_{e(1)_\phi} e(1)_\phi \rangle_{g_1} = 0, \quad \overset{1}{C}_{\phi\phi A} = \left\langle e(1)_\phi, \overset{1}{\nabla}_{e(1)_\phi} e(1)_A \right\rangle_{g_1} = (1/2)\frac{\partial \ln f_1}{\partial x^A},$$

$$\overset{1}{C}_{\phi AB} = \left\langle e(1)_\phi, \overset{1}{\nabla}_{e(1)_A} e(1)_B \right\rangle_{g_1} = 0, \quad \overset{1}{C}_{A\phi B} = \left\langle e(1)_A, \overset{1}{\nabla}_{e(1)_\phi} e(1)_B \right\rangle_{g_1} = 0,$$

$$\overset{1}{C}_{A\phi\phi} = \left\langle e(1)_A, \overset{1}{\nabla}_{e(1)_\phi} e(1)_\phi \right\rangle_{g_1} = -(1/2)\frac{\partial \ln f_1}{\partial x^A}, \quad \overset{1}{C}_{AB\phi} = \left\langle e(1)_A, \overset{1}{\nabla}_{e(1)_B} e(1)_\phi \right\rangle_{g_1} = 0,$$

$$\overset{1}{C}_{ABC} = \left\langle e(1)_A, \overset{1}{\nabla}_{e(1)_B} e(1)_C \right\rangle_{g_1} = \Lambda_{ABC} + \overline{G}_{AC}\frac{\partial}{\partial x^B}.$$

We get $\overset{2}{C}$ replacing f_1 in expressions for $\overset{1}{C}$ by qf_1.

$$\overset{2}{C}_{\phi\phi\phi} = 0 = \overset{2}{C}_{\phi AB} = \overset{2}{C}_{AB\phi},\ \overset{2}{C}_{\phi\phi A} = (1/2)\frac{\partial \ln (qf_1)}{\partial x^A} = -\overset{2}{C}_{A\phi\phi},\ \overset{2}{C}_{ABC} = \overset{1}{C}_{ABC}$$

In the following Clifford multiplication by $e(1)^i$and $e(2)^i$ are denoted by $\cdot$. Distinction is not necessary because it is multiplication by the same matrix. Clifford relation is $e(1)^i \cdot e(1)^k + e(1)^k \cdot e(1)^i = -2\delta^{ij}$. For the $SU(2)$ spinor $\xi \in \mathbb{C}^2$, using $\overset{1}{\nabla}_{e(1)_k}\xi = e(1)_k(\xi) - \frac{1}{4}\left\langle e(1)_i, \overset{1}{\nabla}_{e(1)_k} e(1)_j\right\rangle_{g_1} e(1)^i \cdot e(1)^j \cdot \xi$, we get

$$\begin{aligned}
\overset{1}{\nabla}_{e(1)_B}\xi &= e(1)_B(\xi) - \frac{1}{4}\overset{1}{C}_{ijB} e(1)^i \cdot e(1)^j \cdot \xi \\
&= e(2)_B(\xi) + (1/4)\overset{1}{C}_{\phi\phi B}\xi - (1/4)\overset{1}{C}_{ACB} e(1)^A \cdot e(1)^C \cdot \xi \\
&= e(2)_B(\xi) + \frac{1}{4}\overset{2}{C}_{\phi\phi B}\xi - (1/4)\overset{2}{C}_{ACB} e(2)^A \cdot e(2)^C \cdot \xi - (1/8)\left(\frac{\partial \ln q}{\partial x^B}\right)\xi \\
&= \overset{2}{\nabla}_{e(2)_B}\xi - (1/8)\left(\frac{\partial \ln q}{\partial x^B}\right)\xi
\end{aligned}$$

Similarly for $e(2)_\phi(\xi) = 0 = e(1)_\phi(\xi)$,

$$\begin{aligned}
\overset{1}{\nabla}_{e(1)_\phi}\xi &= e(1)_\phi(\xi) - (1/4)\overset{1}{C}_{ij\phi} e(1)^i \cdot e(1)^j \cdot \xi = (1/4)\left(\frac{\partial \ln f_1}{\partial x^A}\right) e(1)^A \cdot e(1)^\phi \cdot \xi \\
&= (1/4)\left(\frac{\partial \ln (qf_1)}{\partial x^A}\right) e(1)^A \cdot e(1)^\phi \cdot \xi - (1/4)\left(\frac{\partial \ln q}{\partial x^A}\right) e(1)^A \cdot e(1)^\phi \cdot \xi \\
&= \overset{2}{\nabla}_{e(2)_\phi}\xi - (1/4)\left(\frac{\partial \ln q}{\partial x^A}\right) e(2)^A \cdot e(2)^\phi \cdot \xi
\end{aligned}$$

So $D_{g_2}\xi = Dg_1\xi + \frac{3}{8}\left(\frac{\partial \ln q}{\partial x^B}\right) e(2)_B \cdot \xi$ giving $D_{g_2}\left(q^{-\frac{3}{8}}\xi\right) = q^{-\frac{3}{8}} D_{g_1}\xi$. This proves Lemma 8.1.

To prove Lemma 9.2 we also need

$$\overset{2}{\nabla}_{e(2)_\phi}\xi = \frac{1}{4}\left(\frac{\partial \ln (qf_1)}{\partial x^A}\right) e(2)^A \cdot e(2)^\phi \cdot \xi$$

Then

$$\begin{aligned}
\|\overset{1}{\nabla}\xi\|^2 = \|\overset{2}{\nabla}\xi\|^2 + \sum_B \left(-(1/8)\left(\frac{\partial \ln q}{\partial x^B}\right)\left(\left\langle \overset{2}{\nabla}_{e(2)_B}\xi, \xi\right\rangle + \left\langle \xi, \overset{2}{\nabla}_{e(2)_B}\xi\right\rangle\right)\right) + \frac{1}{64}\left(\frac{\partial \ln q}{\partial x^B}\right)\left(\frac{\partial \ln q}{\partial x^B}\right)\|\xi\|^2 \\
- (1/16)\left(\frac{\partial \ln q}{\partial x^A}\right)\left(\frac{\partial \ln (qf_1)}{\partial x^B}\right)\left(\left\langle e(2)^B \cdot e(2)^\phi \cdot \xi, e(2)^A \cdot e(2)^\phi \cdot \xi\right\rangle + \left\langle e(2)^A \cdot e(2)^\phi \cdot \xi, e(2)^B \cdot e(2)^\phi \cdot \xi\right\rangle\right) \\
+ (1/16)\left(\frac{\partial \ln q}{\partial x^A}\right)\left(\frac{\partial \ln q}{\partial x^B}\right)\left\langle e(2)^A \cdot e(2)^\phi \cdot \xi, e(2)^B \cdot e(2)^\phi \cdot \xi\right\rangle \qquad (130)
\end{aligned}$$

Now $\left\langle e(2)^B \cdot \xi, e(2)^A \cdot \xi \right\rangle + \left\langle e(2)^A \cdot \xi, e(2)^B \cdot \xi \right\rangle = 2||\xi||^2$. Also $\left\langle e(2)^A \cdot \xi, e(2)^B \cdot \xi \right\rangle =$ $-\left\langle e(2)^B \cdot e(2)^A \cdot \xi, \xi \right\rangle = 2\delta^{AB}||\xi||^2 - \left\langle e(2)^B \cdot \xi, e(2)^A \cdot \xi \right\rangle$. So Eq. (130) gives

$$||\overset{1}{\nabla}\xi||^2 = ||\overset{2}{\nabla}\xi||^2 - \frac{1}{8}\left\langle \overline{\nabla} \ln q, \overline{\nabla}||\xi||^2 \right\rangle_G - \frac{3}{64}|\overline{\nabla} \ln q|_G^2 ||\xi||^2 - \frac{1}{8}\left\langle \overline{\nabla} \ln q, \overline{\nabla} \ln f_1 \right\rangle_G ||\xi||^2 \tag{131}$$

We take

$$\begin{aligned} g_1 &= \chi = \sigma^2 \zeta \overline{g} + U f d\phi^2 \\ g_2 &= \gamma = \sigma^2 \zeta \overline{g} + |U|^{-4} U f d\phi^2 \\ f_1 &= U f, \qquad q = |U|^{-4} \end{aligned}$$

Then Eq. (131) gives $||\nabla_\chi \Theta_\gamma||^2 = ||\nabla_\gamma \Theta_\gamma||^2 + (1/2)\sigma^{-2}\zeta^{-1}\left\langle \overline{\nabla} \ln |U|, \overline{\nabla}||\Theta_\gamma||^2 \right\rangle - (3/4)\sigma^{-2}\zeta^{-1}\left\langle \overline{\nabla} \ln |U|, \overline{\nabla} \ln |U| \right\rangle ||\Theta_\gamma||^2 +$ $(1/2)\sigma^{-2}\zeta^{-1}\left\langle \overline{\nabla} \ln |U|, \overline{\nabla} \ln(U f) \right\rangle ||\Theta_\gamma||^2$ where norms of vectors or forms and inner product of vectors and forms are with respect to 2-metric $\overline{g}$.

We have $\Theta_\gamma = |U|^{1/2}\Theta_\chi$. Recalling $\overset{1}{\nabla} = \nabla_\chi$ we get,
$||\nabla_\chi \Theta_\gamma||^2 = ||\nabla_\chi \left(|U|^{1/2}\Theta_\chi \right)||^2 = ||(1/2)|U|^{1/2} \left(\nabla \ln |U| \right) \Theta_\chi + |U|^{1/2}\nabla_\chi \Theta_\chi||^2$
$= (1/4)\sigma^{-2}\zeta^{-1}\left\langle \overline{\nabla} \ln |U|, \overline{\nabla} \ln |U| \right\rangle ||\Theta_\gamma||^2 + |U|||\nabla_\chi \Theta_\chi||^2 + (1/2)|U|\left\langle \nabla \ln |U|, \nabla||\Theta_\chi||^2 \right\rangle_\chi$.
Thus for some pure imaginary function (or zero) Im we get

$$\begin{aligned} &(1/4)\sigma^{-2}\zeta^{-1}\left\langle \overline{\nabla} \ln |U|, \overline{\nabla} \ln |U| \right\rangle ||\Theta_\gamma||^2 + |U|||\nabla_\chi \Theta_\chi||^2 + (1/2)|U|\left\langle \nabla \ln |U|, \nabla||\Theta_\chi||^2 \right\rangle_\chi + Im \\ &= ||\nabla_\gamma \Theta_\gamma||^2 + (1/2)\sigma^{-2}\zeta^{-1}\left\langle \overline{\nabla} \ln |U|, \overline{\nabla}||\Theta_\gamma||^2 \right\rangle - (1/4)\sigma^{-2}\zeta^{-1}\left\langle \overline{\nabla} \ln |U|, \overline{\nabla} \ln |U| \right\rangle ||\Theta_\gamma||^2 \\ &\qquad + (1/2)\sigma^{-2}\zeta^{-1}\left\langle \overline{\nabla} \ln |U|, \overline{\nabla} \ln f \right\rangle ||\Theta_\gamma||^2 \end{aligned}$$

Now since $||\Theta_\chi||^2 = |U|^{-1}||\Theta_\gamma||^2$, $\quad \overline{\nabla}||\Theta_\chi||^2 = -|U|^{-1}||\Theta_\gamma||^2\overline{\nabla} \ln |U| + |U|^{-1}\overline{\nabla}||\Theta_\gamma||^2$ which gives

$$\left\langle \overline{\nabla} \ln |U|, \overline{\nabla}||\Theta_\chi||^2 \right\rangle = -||\Theta_\chi||^2 \left\langle \overline{\nabla} \ln |U|, \overline{\nabla} \ln |U| \right\rangle + |U|^{-1}\left\langle \overline{\nabla} \ln |U|, \overline{\nabla}||\Theta_\gamma||^2 \right\rangle$$

Thus we get

$$|U|||\nabla_\chi \Theta_\chi||^2 + Im = ||\nabla_\gamma \Theta_\gamma||^2 + (1/2)\sigma^{-2}\zeta^{-1}\left\langle \overline{\nabla} \ln |U|, \overline{\nabla} \ln f \right\rangle ||\Theta_\gamma||^2$$

which gives Lemma 9.2.

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